Math 7382: Qualifying Exam Syllabus

This core course covers four main topics: tensor algebra and calculus, continuum mechanics, Fourier analysis, and weak-form differential equations. The first of these is by way of reviewing and establishing elementary tools from vector calculus and linear algebra, which are used in three subsequent topics. They might not be tested specifically on the qualifier exam, but the techniques appear prominently in the problems.

The problems in the test bank are marked according to their pertinence to three categories: (1) continuum mechanics, (2) Fourier analysis, and (3) distributions and weak-form differential equations. Some problems pertain to multiple categories. More specifically, the three categories cover the following material.

1. Equations of Continuum Physics
   (a) Fluid mechanics and acoustics
   (b) Solid mechanics: elasticity
   (c) Electricity and Magnetism
   (d) Helmholtz decomposition
2. Fourier Analysis
   (a) Fourier series
   (b) Fourier transform
   (c) Classical solutions to PDE
   (d) Initial-value problems
   (e) Sobolev embedding theorem
3. Weak-form Differential Equations
   (a) Hilbert space
   (b) Distributions
   (c) Weak-form PDE

Math 7382: Qualifying Exam Problem Bank

1. (Category: 3)
Let $\Omega \subset \mathbb{R}^3$ be bounded with C^1 boundary.

a. Suppose $u, v \in C^2(\mathbb{R}^3)$. Show

$$\int_\Omega (u\triangle v - v\triangle u)dV = \int_{\partial\Omega} \Big(u\nabla v - v\nabla u\Big) \cdot n\, dS$$

where n is the outward facing normal to the surface $\partial\Omega$.

b. For $y \in \Omega$, suppose there exists $G_y \in C^1(\overline{\Omega} \setminus \{y\})$ such that $\triangle G_y = \delta_y$ in the sense of distributions. For $\phi \in C_c^\infty(\Omega)$ a test function, δ_y is defined by $\langle \delta_y, \phi\rangle = \phi(y)$. If $v \in C^2(\mathbb{R}^3)$, show

$$v(y) = \int_{\partial\Omega} (G_y(x)\nabla v(x) - \nabla G_y(x)v(x)) \cdot n\, dS_x.$$

2. (Category: 1, 3)

a. Prove the identity for $\Phi, \Psi \in C^1(\mathbb{R}^3; \mathbb{R}^3)$:

$$\nabla \cdot (\Psi \times \Phi) = \Phi \cdot (\nabla \times \Psi) - (\nabla \times \Phi) \cdot \Psi.$$

b. Consider simply connected open domain $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$, where Ω_1 and Ω_2 are disjoint simply connected open sets. Let Γ be the boundary between Ω_1 and Ω_2. Consider the curl operator in the sense of distributions denoted $\overline{\nabla}\times$ on $C_c^\infty(\Omega; \mathbb{R}^3)$. Let F be defined such that $F(x) = F_j(x)$ for $x \in \overline{\Omega}_j$ where $F_j \in C^2(\overline{\Omega_2}; \mathbb{R}^3)$. Let $[F](x) := F_2(x) - F_1(x)$ for $x \in \Gamma$. Let $N(x)$ be the normal vector for $x \in \Gamma$ aiming into Ω_2. Prove

$$\overline{\nabla} \times F = \nabla \times F_1\Big|_{\Omega_1} + \nabla \times F_2\Big|_{\Omega_2} + N \times [F]\delta_\Gamma.$$

For a function $G(x)$, the distribution $G\delta_\Gamma$ is defined such that $\langle G\delta_\Gamma, \Phi\rangle = \int_\Gamma G \cdot \Phi$.

3. (Category: 2)
Consider the PDE $-\triangle u + \gamma u = f$ for $f \in \mathcal{S}(\mathbb{R}^3)$ and $\gamma > 0$ a scalar.

a. Compute the solution $u(x)$ using Fourier analysis *(no need to prove it is a solution in this part).*

b. Prove $u \in \mathcal{S}(\mathbb{R}^d)$, and conclude it is a classical solution to the PDE. *Hint: if* $v \in \mathcal{S}(\mathbb{R}^3)$, *then* $pv \in \mathcal{S}(\mathbb{R}^3)$ *for any polynomial* p.

4. (Category: 1,2)
Let $\mathcal{V} = \mathbb{R}^3$. Consider the linear elastic equation

$$\rho_0 \frac{\partial^2}{\partial t^2} u = \nabla \cdot [A(\nabla u)] \tag{1}$$

for $A \in \mathcal{V}^4$ and u a smooth vector field and $\rho_0 > 0$ a constant. Suppose if $H \in \mathcal{V}^2$ and $\lambda, \mu > 0$,

$$A(H) = \lambda(\mathrm{tr}\ H)I + 2\mu\ \mathrm{Sym}(H). \tag{2}$$

a. Derive the Navier equation

$$\rho_0 \frac{\partial^2}{\partial t^2} u = \mu \triangle u + (\lambda + \mu)\nabla(\nabla \cdot u).$$

b. Assume the PDE is defined over $\mathbb{R}^3$. Derive the solution for $u(x, t)$ using Fourier analysis **(no need to prove it is a classical solution, you're only asked to calculate)**. Assume $u(x, 0) = f(x)$ and $\partial_t u(x, 0) = g(x)$ for f and g smooth and compactly supported vector fields.

5. (Category: 1)
Suppose $B = \mathbb{R}^3$ and the body force is 0. Let $\{e_i\}_i$ be an orthonormal coordinate frame. *In this problem, assume all the initial value problems discussed have solutions, and all fields are smooth.* Consider the Cauchy stress tensor corresponding to *pure shear stress* given by

$$S(x, t) = f \circ \rho(x, t)\mathcal{P}, \qquad \mathcal{P} = e_1 \otimes e_2 + e_2 \otimes e_1.$$

Here $\circ$ is function composition, and $f(\cdot)$ is scalar-valued.

a. Find a family of disjoint planes $\Gamma_\lambda \subset B$ for $\lambda \in \mathbb{R}$ such that $B = \cup_\lambda \Gamma_\lambda$ with normal field denoted $n_\lambda(x)$ for $x \in \Gamma_\lambda$ such that $S(x, t)n_\lambda(x) = 0$ for all $x \in \Gamma_\lambda$.

b. Suppose the body force is 0 and initial velocity is zero such that the continuum dynamics is given by the initial value problem in spatial coordinates

$$\begin{aligned} &\partial_t \rho(x, t) + \nabla^x \cdot (\rho(x, t)v(x, t)) = 0, & \rho(x, t)\frac{d}{dt}v(x, t) &= \nabla^x \cdot S(x, t), \\ &\rho(x, 0) = \rho_0(x), & v(x, 0) &= 0 \end{aligned}$$

where for flow map $\varphi(X,t)$ over material coordinates, we denote spatial coordinates $x = \varphi(X,t)$ with velocity field $v(x,t) = \frac{d}{dt}x = \partial_t \varphi(X,t)$.
Find $L_\lambda : \mathbb{R}^2 \to \Gamma_\lambda$ parameterization of the surface Γ_λ, and suppose

$$p_\lambda(y,t) := \rho(L_\lambda y, t), \qquad \nu_\lambda(y,t) := v(L_\lambda y, t). \tag{3}$$

Find an initial value problem in p_λ and ν_λ, and verify that solutions to your choice of family of PDEs in $\mathbb{R}^2$ combine via (3) to yield solutions to the initial value problem over the whole body given above. *This is an example of dimensional reduction.*

6. (Category: 1)
For deformation map $\varphi(X,t)$, we define $F(X,t) = \nabla^X \varphi(X,t)$, spatial coordinates where $x = \varphi(X,t)$, and spatial velocity field $v(x,t) = \partial_t \varphi(X,t)$.

a. Prove
$$\frac{\partial}{\partial t} \det F(X,t) = \det F(X,t) (\nabla^x \cdot v)(x,t) \Big|_{x=\varphi(X,t)}.$$

b. Prove
$$\frac{d}{dt} v(x,t) = \frac{\partial}{\partial t} v(x,t) + \left(\nabla^x v(x,t)\right) v(x,t)$$

where $\frac{d}{dt}$ is understood as the partial time derivative of *fields in material coordinates* (X,t).

7. (Category: 1)
Define a body B_t with density field $\rho(x,t)$, Cauchy stress tensor $S(x,t)$, and body force $\rho(x,t)b(x,t)$ written as spatial fields. Assume all fields are smooth. Consider kinetic energy of $\Omega_t \subset B_t$ defined as
$$K[\Omega_t] = \int_{\Omega_t} \frac{1}{2} \rho(x,t)\, v(x,t) \cdot v(x,t) dV_x.$$

Consider the following balance equations in spatial coordinates

$$\rho \frac{d}{dt} v = \nabla^x \cdot S + \rho b, \quad \partial_t \rho + \nabla^x \cdot (\rho v) = 0, \quad S = S^T.$$

Here $\frac{d}{dt}$ is understood as the partial derivative in time *in material coordinates.*

a. Prove
$$\nabla^x \cdot (S^T v) = (\nabla^x \cdot S) \cdot v + S : \nabla^x v.$$

b. Prove that for any $\Omega_t = \varphi_t(\Omega)$, $\Omega \subset B$ open, then

$$\int_{\Omega_t} \rho v \cdot \frac{d}{dt} v \, dV_x + \int_{\Omega_t} S : \mathrm{sym}(\nabla^x v) \, dV_x = \int_{\partial\Omega_t} v \cdot Sn \, dA_x + \int_{\Omega_t} \rho b \cdot v \, dV_x$$

8. (Category: 1)

a. Suppose $v \in C^\infty(\mathbb{R}^3; \mathcal{V})$ such that there exists $\psi \in C^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$ where

$$v(x,t) = \nabla^\perp \psi(x_1, x_2, t) = \partial_2 \psi(x_1, x_2, t) e_1 - \partial_1 \psi(x_1, x_2, t) e_2.$$

Show there is a scalar field $f(x_1, x_2, t)$ such that

$$\nabla^x \times v = f e_3, \qquad f = -\triangle^x \psi.$$

b. Consider the balance of linear momentum equation for Navier-Stokes with no body force (assume p smooth):

$$\rho_0(\partial_t v + (\nabla^x v) v) = \mu \triangle^x v - \nabla^x p.$$

We denote $\nabla^x f = \partial_{x_1} f e_1 + \partial_{x_2} f e_2$. Derive

$$\partial_t f + v \cdot \nabla^x f = \frac{\mu}{\rho_0} \triangle^x f$$

9. (Category: 1)
Let the vector fields E and H in $\mathbb{R}^3$ (with coordinates (x_1, x_2, x_3)) satisfy the free harmonic Maxwell system

$$\nabla \times E = i\omega\mu H \tag{4}$$
$$\nabla \times H = -i\omega\epsilon E \tag{5}$$

Suppose that E and H are independent of x_3, that H is perpendicular to the (x_1, x_2)-plane, and that ϵ and μ are smooth scalar functions of x_1 and x_2 alone, and $\epsilon(x_1, x_2) \neq 0$. Show that the Maxwell system can be reduced to a single scalar second-order PDE for H.

10. (Category: 1)

Consider an invertible smooth map

$$\varphi : (0,1)^3 \to W \subset \mathbb{R}^3$$

satisfying $\det \nabla^X \varphi(X) > 0$. Consider the set of integers $\mathbb{Z}/N = \{1, 2, \cdots N\}$. Let $G_N = (\mathbb{Z}/N)^3$ and define

$$P_N = \{(\frac{i_1}{N}, \frac{i_2}{N}, \frac{i_3}{N}) : (i_1, i_2, i_3) \in G_N\},$$

a discretization of the cube. This can be considered a *reference configuration*. Suppose a point charge is placed at each point with charge $q_N = \frac{Q}{N^3}$ for Q constant. Then we define a distribution $\rho_N \in \mathcal{D}'(W)$ corresponding to charge density given by

$$\rho_N := \sum_{p \in P_N} q_N \, \delta_{\varphi(p)}$$

where $\langle \delta_x, \phi \rangle := \phi(x)$ for $\phi \in C_c^\infty(W)$. Show $\rho_N \to \langle \rho, \cdot \rangle \in \mathcal{D}'$ for some $\rho \in C(W)$. Find ρ.

11. (Category: 2)
Consider the Fejér Kernel given by $\{F_N\}_{N=1}^\infty$ where

$$F_N(x) = \frac{1}{N} \sum_{j=0}^{N-1} D_j(x), \qquad D_j(x) = \sum_{n=-N}^{N} e^{-2\pi i x n}.$$

Prove that the Fejér Kernel is a good kernel, and

$$F_N(x) = \frac{1}{N} \frac{\sin^2(N\pi x)}{\sin^2(\pi x)}.$$

12. (Category: 2)

a. Suppose $f \in C^{(k)}_{\text{per}}([-1/2, 1/2])$, i.e. k-times continuously differentiable and periodic. Recall

$$\hat{f}(n) = \langle f, \phi_n \rangle, \qquad \phi_n(x) = e^{-2\pi i x n}.$$

Show for $n \neq 0$ that there exists a constant $C > 0$ such that

$$|\hat{f}(n)| \leq C|n|^{-k}.$$

b. Prove if $k > 1$ that the partial Fourier sum $S_N(f) \to f$ pointwise uniformly on $x \in [-1/2, 1/2]$.

13. (Category: 2)
Suppose $g \in C(\mathbb{R})$ be periodic with period 1, i.e. $g(x+1) = g(x)$ $\forall x \in \mathbb{R}$. Let $\alpha \in \mathbb{R}$ be irrational. Define the sequence

$$G_N = \frac{1}{2N+1} \sum_{n=-N}^{N} g(\alpha n).$$

Prove

$$G_N \to \int_{-1/2}^{1/2} g(x)dx \qquad \text{as } N \to \infty.$$

14. (Category: 2)
For $1 \le p < \infty$, the discrete ℓ^p space is defined as

$$\ell^p(\mathbb{Z}^3) = \{g : \mathbb{Z}^3 \to \mathbb{C} : \|g\|_p = \Big(\sum_{n \in \mathbb{Z}^3} |g(n)|^p \Big)^{1/p} < \infty\}.$$

Consider $\mathfrak{F} : L^2([-1/2, 1/2]^3) \to \ell^2(\mathbb{Z}^3)$ be defined such that

$$\mathfrak{F}f(n) = \langle f, \phi_n \rangle, \qquad \phi_n(x) = e^{-2\pi n \cdot x}.$$

The Plancherel identity gives

$$\|\mathfrak{F}f\|_2 = \|f\|_2$$

where on the left we mean the discrete-ℓ^2 norm and on the right the continuous L^2 norm. The discrete inner product for ℓ^2 is given by

$$\langle a, b \rangle = \sum_{n \in \mathbb{Z}^3} a_n \bar{b}_n, \qquad a, b \in \ell^2(\mathbb{Z}^3).$$

a. Show if $f, g \in L^2([-1/2, 1/2]^3)$ that

$$\langle \mathfrak{F}f, \mathfrak{F}g \rangle = \langle f, g \rangle.$$

On the left-hand side we mean the discrete ℓ^2 inner product, and on the right the continuous L^2 inner product.

b. We define the discrete convolution as follows: if $a, b \in \ell^1(\mathbb{Z}^3)$, then

$$a * b(n) = \sum_{m \in \mathbb{Z}^3} a(n-m)b(m).$$

Show $a * b \in \ell^1(\mathbb{Z}^3)$.

c. Suppose $f, g \in L^2([-1/2, 1/2]^3)$. Show

$$\mathfrak{F}(fg) = \mathfrak{F}f * \mathfrak{F}g \in \ell^\infty(\mathbb{Z}^3).$$

15. (Category: 2)
Consider the isotropic linear elasticity elastostatics equations for the displacement field u a vector field (a vector field):

$$Lu := \mu \triangle^X u + (\lambda + \mu)\nabla^X(\nabla^X \cdot u) = f := -\rho_0 b_m.$$

Consider the Fourier modes ϕ_n for $n \in \mathbb{Z}^3$.

a. Show for any $q \in \mathbb{R}^3$ that

$$-L(q\phi_n) = \Lambda_n q \phi_n$$

for some $\Lambda_n \in \mathbb{R}^{3\times 3}$. Find Λ_n and verify if $n \neq 0$ that Λ_n is positive definite, i.e. if $q \in \mathbb{R}^3$ is non-zero, $q \cdot (\Lambda_n q) > 0$.

b. Show that if $f \in C^k_{\text{per}}([-1/2, 1/2]^3)$ that for some $C > 0$ we have

$$|\hat{f}(n)| \leq C \frac{1}{|n|^k}, \qquad n \neq 0.$$

where $\hat{f}(n) = \langle f, \phi_n \rangle$, $n \in \mathbb{Z}^3$.

c. If $f \in C^\infty_{\text{per}}([-1/2, 1/2]^3; \mathbb{R}^3)$, find the classical solution to

$$Lu = f$$

when $\hat{f}(0) = 0$. Prove the solution is classical.

16. (Category: 2)
Consider the heat equation $\frac{\partial}{\partial t}u(x,t) = \triangle u(x,t)$, and suppose $u(x,0) = u_0(x)$ where $u_0 \in \mathcal{S}(\mathbb{R}^d)$. Here $u(x,t)$ is a scalar field.

a. Find the classical solution $u(x,t)$ and verify that it is a classical solution, i.e. that u is twice continuously differentiable in x and continuously differentiable in t, satisfies the PDE and initial data.

b. Now suppose $u_0 \in L^2(\mathbb{R}^d)$. Use the same formulation for your solution as in part (a) and show that $u(x,t)$ satisfies classically the PDE $\frac{\partial}{\partial t}u(x,t) = \triangle u(x,t)$ for $t > 0$.

17. (Category: 2)
Consider the inhomogeneous Helmholtz equation

$$(1 - \triangle)u(x) = f(x) \tag{6}$$

for $x \in \mathbb{R}^d$.

a. Suppose $f \in \mathcal{S}(\mathbb{R}^d)$. Find $u(x)$ that solves the PDE such that $u(x) \to 0$ as $|x| \to \infty$.

b. Suppose $f_n \in \mathcal{S}(\mathbb{R}^d)$ and $f_n \to f$ in the $L^2(\mathbb{R}^d)$ norm sense (i.e. $\|f_n - f\|_2 \to 0$). Suppose u_n is a solution to

$$(1 - \triangle)u_n(x) = f_n(x).$$

Show $u_n \to u$ in the L^2 sense, and show u satisfies

$$\langle (1 - \triangle)\psi, u \rangle = \langle \psi, f \rangle$$

for all $\psi \in \mathcal{S}(\mathbb{R}^d)$.

18. (Category: 2)
Suppose $g \in C(\mathbb{R})$ real-valued and bounded. Define the operator $g(-\triangle)$ acting on $\psi \in \mathcal{S}(\mathbb{R}^d)$ by

$$g(-\triangle)\psi(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} g(4\pi|\xi|^2)\hat{\psi}(\xi)d\xi.$$

a. Suppose $f \in C(\mathbb{R})$ real-valued and bounded. Show $f(-\triangle)g(-\triangle)\psi = (fg)(-\triangle)\psi$.

b. Show $f(-\triangle) + cg(-\triangle) = (f + cg)(-\triangle)$ for c a constant.

c. Show if $p(x) = \sum_{j=0}^{n} a_j x^j$ is a polynomial that

$$p(-\triangle)\psi(x) = \sum_{j=0}^{n} a_j(-\triangle)^j \psi(x)$$

where the left-hand side is defined as above and $(-\triangle)^j$ is simply the laplacian applied j times.

19. (Category: 2)
Consider $\sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ smooth. Suppose we define an operator Q acting on $\mathcal{S}(\mathbb{R}^d)$ by

$$Q\psi(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \sigma(x,\xi)\psi(y)e^{2\pi i(x-y)\cdot\xi} dy d\xi. \tag{7}$$

a. Show that if $\sigma(x,\xi) = q(x) + \xi^T A \xi$ for A a $d \times d$ positive definite matrix that Q can be written as a linear differential operator. Find that linear differential operator.

b. Suppose $Q\psi(x) = g(x) \cdot \nabla\psi(x)$ for some vector-valued continuous and bounded function $g(x)$. Find $\sigma(x,\xi)$ that satisfies (7).

20. (Category: 2)
Consider the PDE $u : \mathcal{D} = [0,\infty) \times [0,1]^2 \times [0,\infty) \to \mathbb{R}$:

$$\begin{aligned}
&\nabla \cdot A\nabla u - \gamma u = \frac{\partial^2}{\partial t^2} u, \\
&u(x,y,z,t) = 0 \text{ for } (y,z) \in \partial([0,1]^2) \text{ and all } t \geq 0, \\
&u(0,y,z,t) = f(0,y,z) \text{ for all } t \geq 0, \\
&u(x,y,z,0) = f(x,y,z).
\end{aligned}$$

Here A is a self-adjoint positive-definite 3×3 matrix, $\gamma > 0$, and $f \in \mathcal{S}([0,\infty) \times [0,1]^2)$ with $f(x,y,z) = 0$ for $(y,z) \in \partial([0,1]^2)$. Derive a formula for $u(x,y,z,t)$ using Fourier analysis.

21. (Category: 3)
Let Γ be a smooth surface in $\mathbb{R}^3$, such that Γ divides $\mathbb{R}^3$ into two disjoint open regions, Ω_1 and Ω_2, with $\mathbb{R}^3 = \Omega_1 \cup \Omega_2 \cup \Gamma$. Let n be the normal vector at each point of Γ, directed

into Ω_2. Let F be a smooth vector field in $\Omega_1 \cup \Omega_2$ that is smoothly extensible from Ω_1 to a vector field F_1 in $\Omega_1 \cup \Gamma$ and from Ω_2 to a vector field F_2 in $\Omega_2 \cup \Gamma$; and denote the jump in F across Γ at each $x \in \Gamma$ by

$$[F](x) \;=\; F_2(x) - F_1(x).$$

For any function g integrable on Γ, denote by $g\delta_\Gamma$ the distribution defined by

$$\langle g\delta_\Gamma, \phi\rangle \;=\; \int_\Gamma g\phi \qquad \forall \phi \in C_c^\infty.$$

Consider F to be a distribution, and denote its divergence *as a distribution* by $\overline{\nabla}\cdot F$. Prove that

$$\overline{\nabla}\cdot F \;=\; \nabla\cdot F_1|_{\Omega_1} + \nabla\cdot F_2|_{\Omega_2} + [F]\cdot n\,\delta_\Gamma.$$

22. (Category: 3)
Find the distributional derivative of $\ln|x|$ in $\mathcal{D}'(\mathbb{R}^d)$.

23. (Category: 3)
Prove that the Dirac delta-function is not equal as a distribution to any continuous function.

24. (Category: 3)
This problem shows that multiplication of distributions is not a continuous operation (typically multiplication of distributions is not defined anyway). Prove that

$$\lim_{n\to\infty} \sin(nx) \;=\; 0$$

in $\mathcal{D}'(\mathbb{R})$, but that

$$\lim_{n\to\infty} \sin^2(nx) \;\neq\; 0$$

25. (Category: 3)
Suppose $\Omega \subset \mathbb{R}^3$ is a bounded open set with Lipschitz continuous boundary. Consider the operator $L : C_c^\infty(\Omega;\mathbb{C}^3) \to C_c^\infty(\Omega;\mathbb{C}^3)$ defined by

$$L\psi = \mu\triangle\psi + (\lambda+\mu)\nabla(\nabla\cdot\psi).$$

Prove $Lu = f$ has a weak solution, i.e. a distributional solution $u \in L^1_{\text{loc}}(\Omega; \mathbb{C}^3)$, for $f \in L^2(\Omega; \mathbb{C}^3)$ with respect to the test function space $C_c^\infty(\Omega; \mathbb{C}^3)$. Show $u \in L^2(\Omega)$.

26. (Category: 3)
Let $\Omega \subset \mathbb{R}^3$ be bounded and open and $0 < \tau_1 < \tau(x) < \tau_2$. Consider the PDE in the weak sense, $\nabla \cdot A(x)\nabla u(x) + \lambda\tau(x)u(x) = f(x)$ for $f \in L^2(\Omega)$ where $A(x) \in \mathbb{R}^{3\times 3}$ is self-adjoint, positive definite, and bounded.
Show there exists a weak solution for all λ except possibly a countable set in $H_0^1(\Omega)$, which is the closure of $C_c^\infty(\Omega)$, with respect to the norm

$$\|\psi\|_1^2 := \int_\Omega \nabla\psi(x) \cdot A(x)\overline{\nabla\psi(x)}dx.$$

27. (Category: 3)
Consider the weak formulation of the PDE

$$\nabla \cdot \tau(x)\nabla u \;=\; f,$$

when the material coefficient $\tau(x)$ takes on different constant values on two different components of an object.
Let Ω be a bounded open set in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$ and closure $\overline{\Omega}$. Let Γ be a smooth hypersurface in $\mathbb{R}^d$ that divides $\mathbb{R}^d$ into two disjoint open "halves", W_- and W_+, and let n denote the normal vector to Γ that points into W_+ or the normal vector to $\partial\Omega$ that points out of Ω. Denote the intersections of these halves with Ω by $\Omega_- = W_- \cap \Omega$ and $\Omega_+ = W_+ \cap \Omega$. Let $\tau(x)$ be piecewise constant:

$$\tau(x) = \begin{cases} \tau_- & \text{for } x \in W_- \\ \tau_+ & \text{for } x \in W_+ \end{cases}$$

and assume that f is continuous in Ω_- and in Ω_+.
Let $u : \overline{\Omega} \to \mathbb{C}$ be a continuous function that is twice differentiable in each of Ω_- and Ω_+. Let u be equal to zero on the boundary $\partial\Omega$. Prove that

$$\int_\Omega \tau(x)\nabla u \cdot \nabla\phi \;=\; \int_\Omega f\phi \qquad \forall\phi \in C_c^\infty(\Omega)$$

if and only if $\forall x \in \Omega$,

$$\begin{aligned} \tau_-\nabla^2 u(x) &= -f(x) && \text{if } x \in \Omega_- \\ \tau_+\nabla^2 u(x) &= -f(x) && \text{if } x \in \Omega_+ \\ \tau_-\nabla u_-(x) \cdot n = \tau_+&\nabla u_+(x) \cdot n && \text{if } x \in \Gamma \end{aligned}$$

in which $\nabla u_{\pm}(x)$ indicates the limit of $\nabla u(x)$ to Γ from inside $\Omega_{\pm}$.

28. (Category: 1)
Consider a body in material coordinates $B \subset \mathbb{R}^3$ open. Let $x = \varphi(X,t) = \varphi_t(X)$ be our change from material to spatial coordinates, and let $\rho(x,t)$ define the density field in spatial coordinates. Assume conservation of mass holds. Prove that if $\Phi(x,t)$ is smooth and scalar-valued and $\Omega_t = \varphi_t(\Omega)$ is the time evolution of some open $\Omega \subset B$ that

$$\frac{d}{dt}\int_{\Omega_t} \Phi(x,t)\rho(x,t)dV_x = \int_{\Omega_t} \frac{d}{dt}\Phi(x,t)\ \rho(x,t)dV_x.$$

Here we denote $\frac{d}{dt}\Phi(x,t) := \frac{\partial}{\partial t}\Phi(\varphi(X,t),t)\Big|_{X=\varphi_t^{-1}(x)}$.

29. (Category: 1)
Suppose we had a system with a fixed uniform rate of mass decay described as follows. Let $x = \varphi(X,t) = \varphi_t(X)$ be the map from material coordinates to spatial coordinates, and let $\rho(x,t)$ be the spatial mass density field and let $v(x,t) = \frac{\partial}{\partial t}\varphi(X,t)$. Assume all fields are smooth. Suppose for $\Omega \subset B$ open where B is the body and $\Omega_t = \varphi_t(\Omega)$ that we have the mass function

$$\text{mass}[\Omega_t] = \int_{\Omega_t} \rho(x,t)dV_x.$$

Suppose

$$\frac{d}{dt}\text{mass}[\Omega_t] = -\int_{\Omega_t} \gamma(x,t)\rho(x,t)dV_x \tag{8}$$

for some smooth function $\gamma(x,t) > 0$. From the mass relation in (8), derive the PDE in spatial coordinates

$$\frac{\partial}{\partial t}\rho(x,t) + \nabla^x \cdot (\rho(x,t)v(x,t)) = -\gamma\rho(x,t), \qquad x \in B_t, t \geq 0. \tag{9}$$

30. (Category: 1)
Consider a body B and the relation $x = \varphi(X,t) = \varphi_t(X)$ relating spatial and material coordinates. Let $S(x,t)$ be a Cauchy stress tensor satisfying balance of angular momentum,

$v(x,t)$ the velocity spatial field, and $\rho(x,t)$ the mass density spatial field, and $\rho(x,t)b(x,t)$ the body force field. Assume all fields smooth.
Prove that for any $\Omega_t \subset B$ open with smooth boundary, then

$$\int_{\Omega_t} \rho v \cdot \frac{d}{dt} v \, dV_x + \frac{1}{2}\int_{\Omega_t} S : (\nabla^x v + \nabla^x v^T) \, dV_x = \int_{\partial\Omega_t} v \cdot Sn \, dA_x + \int_{\Omega_t} \rho b \cdot v \, dV_x.$$

Here $\frac{d}{dt}$ acting on a spatial field is understood as the partial derivative in time when the field is written in material coordinates.

31. (Category: 3)
Define the Fourier transforms on $\psi \in \mathcal{S}(\mathbb{R}^d)$ by $\mathcal{F}\psi(\xi) = \int e^{-2\pi i x\cdot\xi}\psi(x)dx$.

a. Consider the delta tempered distribution $\delta \in \mathcal{S}'(\mathbb{R}^d)$. Find $\mathcal{F}\delta$.

b. Find $\mathcal{F}(\mathcal{F}\delta)$.