Mathematics Comprehensive Examination
Algebra Core I
January 2001

Directions: Do the first three problems, marked (*), and one additional problem for a total of four. Turn in only these four problems! Start each problem on a new sheet of paper. Put your name and the problem number at the top of every sheet. You will be graded in part on the clarity as well as on the correctness of your response.

Good luck!

1*.

(a) State the structure theorem for finitely generated abelian groups.
(b) Write down *all* abelian groups (up to isomorphism) of order 72.

2*. Let $M = \begin{pmatrix} 7 & 2 \\ 3 & 6 \end{pmatrix}$.

(a) Show that M is similar to a diagonal matrix over the field of rational numbers.
(b) Let F_p be a field of characteristic $p > 0$. For which values of p is M diagonalizable over F?

3*.

(a) Let R be a ring and M an R-module. What does it mean for M to be a *free* R-module?
b) Let R be an integral domain and I a nonzero ideal in R. Prove that I is a free R-module of rank 1 if and only if I is a principal ideal.

4. Let G be a group and let C denote the center of G.
(a) Show that C is a normal subgroup of G.
(b) If G/C is cyclic, prove that G is abelian.

5. Let R be a finite integral domain. Prove that R is a field.

6. Let K be a subfield of a field L, and X an indeterminate over L. Given $f, g \in K[X]$, and given $s, t \in L[X]$ with $fs + gt = 1$, prove there are $s', t' \in K[X]$ with $fs' + gt' = 1$.