Core-1 Algebra Comprehensive Examination
January 2002

Directions. Choose six out of the eight problems below. Start each problem on a new sheet of paper. Put your name and the problem number at the top of every sheet. Hand in ONLY the six problems you want graded. You have 2 and 1/2 hours for this test. Good luck!

(1) Let $G \subset GL_2(\mathbf{R})$ be the subgroup of upper triangular matrices (i.e. matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $ac \neq 0$).

   (a) Show that $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbf{R} \right\}$ is a normal subgroup of G.

   (b) Show that the quotient group G/N is isomorphic to $\mathbf{R}^\times \times \mathbf{R}^\times$.

(2) Find, up to similarity, all linear transformations $T : V \to V$ with characteristic polynomial $P = (X+1)^2 X^3$ on a vector space V of dimension 5

(3) Determine the structure of the abelian group (as in the Structure Theorem for Finitely Generated Abelian Groups) generated by $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, subject to the relations

$$\begin{aligned} 6\mathbf{x} + 12\mathbf{y} + 4\mathbf{z} &= 0 \\ 14\mathbf{x} + 24\mathbf{y} + 8\mathbf{z} &= 0 \\ 4\mathbf{x} + 24\mathbf{y} + 4\mathbf{z} &= 0. \end{aligned}$$

(4) (a) For $A, B \in M_n(\mathbf{C})$, we define $< A, B >= \mathrm{Tr}(AB^*)$. Show that $< \, , \, >$ is a positive-definite hermitian form on $M_n(\mathbf{C})$.

   (b) Let $V = \{A \in M_2(\mathbf{C}) : \mathrm{Tr}(A) = 0\}$. Find an orthonormal basis of V with respect to the hermitian form $< \, , \, >$ defined in (a).

(5) (a) Show that if $\lambda \in \mathbf{C}$ is an eigenvalue of a unitary matrix then $|\lambda| = 1$. (Recall that a matrix $U \in M_n(\mathbf{C})$ is *unitary* if it satisfies $U^*U = I$.)

(b) Let $U \in SO_n(\mathbf{R})$, where n is *odd*. Show that 1 is an eigenvalue of U. (Recall that $SO_n(\mathbf{R})$ denotes the group of matrices $U \in M_n(\mathbf{R})$ satisfying $U^tU = I$ and $\det(U) = 1$.)

(c) Let $U \in SO_3(\mathbf{R})$. Show that every matrix $U \in SO_3(\mathbf{R})$ is conjugate in $SO_3(\mathbf{R})$ to a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

(6) Let R be a commutative ring. There is a standard theorem that says that if R is a field, then $R[X]$ is a PID. Prove the converse: If $R[X]$ is a PID, then R is a field.

(7) Let $\mathbf{F}_3$ be the field with 3 elements.

(a) Show that $f = X^3+2X+2$ and $g = X^3+2X+1$ are irreducible polynomials in $\mathbf{F}_3[X]$.

(b) Show that the ring homomorphism $\phi : \mathbf{F}_3[X] \to \mathbf{F}_3[X]$ defined by $\phi(X) = 2X + 1$ induces a field isomorphism

$$\bar{\phi} : \mathbf{F}_3[X]/(f) \longrightarrow \mathbf{F}_3[X]/(g).$$

(8) Let M be a module over a ring R. Let $P \subset M$ be a submodule and let $\pi : M \to M/P$ be the canonical projection. Show that the following conditions are equivalent:

(i) There exists a submodule $Q \subset M$ such that $M = P \oplus Q$
(ii) There exists a homomorphism $r : M \to P$ such that $r|_P = \mathbf{1}_P$.
(iii) There exists a homomorphism $s : M/P \to M$ with $\pi \circ s = \mathbf{1}_{M/P}$