MATHEMATICS COMPREHENSIVE EXAMINATION

CORE I – Analysis

January 2002

Directions: This test consists of three parts (A), (B), and (C). You must do all ten problems in Part (A), choose two problems from Part (B), and also choose two problems from Part (C). Please answer the problems in the order they appear and turn in only those problems you wish to have graded. You have two and a half hours for this test.

Part (A). Answer "true" or "false". In case "false", give a counterexample.

1. Let $f_n(x) = \begin{cases} n^2 x, & 0 \le x \le 1/n; \\ 1/x, & 1/n < x \le 1. \end{cases}$ The sequence $\{f_n\}$ of functions is pointwise bounded on $[0,1]$.
2. The norm $\|\cdot\|_1$ on ℓ^1 satisfies the parallelogram law: $\|x+y\|_1^2 + \|x-y\|_1^2 = 2\|x\|_1^2 + 2\|y\|_1^2$ for all $x, y \in \ell^1$.
3. If $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$ are measurable sets and $A = \cup_{n=1}^{\infty} A_n$, then 1_{A_n} converges to 1_A in measure. (Here 1_B denotes the indicator function of B, i.e., $1_B(x) = 1$ for $x \in B$ and $= 0$ for $x \notin B$.)
4. If the improper Riemann integral $\int_0^{\infty} f(x)\,dx$ exists, then f must be continuous almost everywhere on $[0,\infty)$.
5. The inequality $\overline{\lim}_{n\to\infty} \int f_n\,d\mu \le \int \overline{\lim}_{n\to\infty} f_n\,d\mu$ holds for any sequence $\{f_n\}$ of nonpositive measurable functions.
6. If $f_n, n \ge 1$, is a sequence of measurable functions converging to f in measure, then it has a subsequence f_{n_k} converging to f almost everywhere.
7. The equality $F(b) = F(a) + \int_a^b F'(x)\,dx$ holds for any uniformly continuous function of bounded variation on $[a,b]$.
8. If f is absolutely continuous on $\mathbb{R}$, then f is a function of bounded variation on $\mathbb{R}$.
9. A linear functional on a normed space X is continuous at every point $x \in X$ if and only if it is continuous at some point $x_0 \in X$.
10. If a Banach space B is separable, then its dual space B^* is also separable.

Part (B). Choose two of the four problems.

B1. Let $P(x)$ be a polynomial with real coefficients. Without referring to any version of the Weierstrass theorem, prove that for any $\varepsilon > 0$ there exists a polynomial $R(x)$ with rational coefficients such that

$$\sup_{0 \leq x \leq 1} |P(x) - R(x)| < \varepsilon.$$

B2. Let $p > r \geq 1$. Assuming the fact that $(a^p + b^p)^{1/p} \leq (a^r + b^r)^{1/r}$ for any $a, b \geq 0$, prove that the inequality

$$(a_1^p + a_2^p + \cdots + a_n^p)^{1/p} \leq (a_1^r + a_2^r + \cdots + a_n^r)^{1/r}$$

holds for any nonnegative numbers $a_1, a_2, \ldots, a_n$.

B3. Let f be a nonnegative continuous function on $[a, b]$ such that the Riemann integral $\int_X f \, d\mu = 0$. Show that f is identically equal to 0.

B4. Let f_n, $n \geq 1$, be a sequence of continuous functions on a metric space (X, d). Suppose f_n converges uniformly to f as $n \to \infty$. Show that f is a continuous function on X.

Part (C). Choose two of the four problems.

C1. Let $\mathcal{A}$ be the collection of all absolutely continuous functions f on $[0, 1]$ such that $\int_0^1 |f'(x)|^2 \, dx \leq 1$. Check whether $\mathcal{A}$ is equicontinuous.

C2. Let $X = \{x_1, x_2, \ldots, x_k, \ldots\}$ be a countable set, $\mathcal{F}$ the collection of all subsets of X, and let μ be the measure on $(X, \mathcal{F})$ such that $\mu(\{x_k\}) = 1/2^k$. Define a sequence $\{f_n\}$ of functions on X by

$$f_n(x_k) = \begin{cases} 0, & \text{if } k = 1, 2, \ldots, n; \\ n, & \text{if } k = n+1, n+2, \ldots. \end{cases}$$

Determine whether the sequence f_n converges in $L^1(\mu)$.

C3. Let $f_n = \frac{1}{n^2} 1_{[0,n]}$, $n \geq 1$. Check whether there is an integrable function g (for the ordinary Lebesgue measure) on the interval $[0, \infty)$ which dominates all functions f_n, $n \geq 1$.

C4. Let f be an integrable function on a measure space (X, μ). Prove that

$$\lim_{n \to \infty} n \cdot \mu\{x \in X; |f(x)| \geq n\} = 0.$$