MATHEMATICS COMPREHENSIVE EXAMINATION

CORE I - ANALYSIS

August 2001

Directions: The test is divided into five sets of problems (A), (B), (C), (D), and (E). Do problem (A) and select one problem from each of the four sets (B) - (E). Please answer each problem on a separate sheet of paper. Turn in only the five problems you wish to have graded. You have 2 and 1/2 hours to complete this test. Good luck!

Problem Set (A). Please answer each problem (a) - (f) with 'true' or 'false' only. Do not explain. Answer the questions (g) - (j) as well as you possibly can.

(a) $M := \{f \in C[0,1] : f$ continuously differentiable and $f'(x) > 0$ for all $x \in [0,1]\}$ is open in the space $C[0,1]$ of real continuous functions on $[0,1]$ equipped with the sup-norm.

(b) Every almost everywhere differentiable function $f : [0,1] \to \mathbb{R}$ is absolutely continuous.

(c) The set of all continuous functions f with $\int_0^1 f(t)\,dt = 0$ is compact in the space $C[0,1]$ of all real continuous functions on $[0,1]$ equipped with the sup norm.

(d) Let $f(0) := 0$ and $f(t) := \frac{1}{\sqrt{t}}$ for $t > 0$. Then f is Lebesgue integrable on $[0,1]$.

(e) A continuous function $f : \mathbb{R} \to \mathbb{R}$ can be approximated uniformly on $\mathbb{R}$ by polynomials.

(f) The set of all functions $f_a : x \to e^{ax}$ with $a \in [0,2]$ is compact in the space $C[0,1]$ of all real continuous functions on $[0,1]$ equipped with the sup norm.

(g) State the fundamental theorem of calculus for (1) continuously differentiable functions and (2) for absolutely continuous functions.

(h) State the Hahn-Banach theorem.

(i) State the open mapping theorem.

(j) State the implicit function theorem.

Problem Set (B).

B.1 Let $f \in L^1(0,\infty)$. Prove or disprove that

(a) $\lim_{n\to\infty} \int_0^n e^{-nx} f(x)\,dx = 0$.
(b) $\lim_{n\to\infty} \int_0^{1/n} f(x)\,dx = 0$.
(c) $\lim_{n\to\infty} \int_n^{\infty} f(x)\,dx = 0$.
(d) For all $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|f(x)| \leq \epsilon$ for almost all $x \geq n$.

B.2 Let $f \in L^1(\mathbb{R})$ (with respect to the Lebesgue measure) such that $\int_{\mathbb{R}} |x||f(x)|\,dx < \infty$. Show that the function $g(y) := \int_{\mathbb{R}} e^{ixy} f(x)\,dx$ is differentiable at every $y \in \mathbb{R}$.

Problem Set (C).

C.1 Let $g_n := n\chi_{[0,n^{-3}]}$ (where $\chi - [a,b]$ denotes the characteristic function of the interval $[a,b]$). Show that $\int_0^1 f(x)g_n(x)\,dx \to 0$ for all $f \in L^2[0,1]$, but not all $f \in L^1[0,1]$.

C.2 Let f be a non-negative Lebesgue measurable function on $(0,\infty)$ such that f^2 is integrable. Let $F(x) := \int_0^x f(t)\,dt$ where $x > 0$. Show that $\lim_{x\to 0+} \frac{F(x)}{\sqrt{x}} = 0$.

Problem Set (D).

D.1 Let

$$f_n(x) = \begin{cases} n^2 x & \text{for } 0 \leq x < 1/n \\ 2n - n^2 x & \text{for } 1/n \leq x \leq 2/n \\ 0 & \text{for } 2/n < x \leq 1. \end{cases}$$

Sketch the graphs of f_1 and f_2. Prove that if g is a continuous real-valued function on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 f_n(x)g(x)dx = g(0).$$

(Hint: First show that $\int_0^1 f_n(x)\,dx = 1$.)

D.2 Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function.

(a) Use Taylor's formula with remainder to show that given x and h then $f'(x) = \frac{f(x+2h)-f(x)}{2h} - hf''(\xi)$ for some ξ.
(b) Assume $f(x) \to 0$ as $x \to \infty$, and that f'' is bounded. Show that $f'(x) \to 0$ as $x \to \infty$.

Problem Set (E).

E.1 (a) Let X be a complete metric space. A mapping $F : X \to X$ is said to be a contraction if there is a constant $r < 1$ such that $d(F(u), F(v)) \leq r \cdot d(u, v)$ for all $u, v \in X$. Given a contraction F and a point $u_0 \in X$, define a sequence $(u_k)_{k \in \mathbb{N}}$ in X by $u_{k+1} = F(u_k)$. Show that $d(u_{k+1}, u_k) \leq r^k d(u_1, u_0)$ and prove that the sequence (u_k) converges to the unique fixed point of F.

(b) Let $g \in C[0,1]$ with $\int_0^1 |g(s)|\, ds \leq r < 1$. Use part (a) to show that, for all $f \in C[0,1]$, there exists a unique solution $u = u(\cdot) \in C[0,1]$ of the equation

$$u(t) = \int_0^t g(t-s)u(s)ds + f(t), \qquad 0 \leq t \leq 1. \tag{$*$}$$

(c) Show that the operator A which assigns to each $f \in C[0,1]$ the unique solution u of the equation $(*)$ is a linear operator from $C[0,1]$ into $C[0,1]$.

(d) Use the 'Closed Graph Theorem' to show that A is a continuous linear operator.

(e) Show that the solutions u of $(*)$ depend continuously on the forcing terms f.

E.2 Let X_0, X_1, X_∞ be the normed linear spaces obtained by putting the norms

$$\|f\|_0 := \sup_{t \in [0,1]} |\int_0^t f(s)\, ds|, \quad \|f\|_1 := \int_0^1 |f(s)|\, ds, \quad \|f\|_\infty := \sup_{t \in [0,1]} |f(t)|$$

on the set of continuous real functions on $[0,1]$.

(a) Show that the normed vector spaces X_0 and X_1 are not complete.

(b) Show that the functions $f_n(t) := \sqrt{n} \sin(nt)$ converge to zero in X_0, but not in X_1 and X_∞.

(c) Show that the linear functional $\Lambda f := f(1/2)$ is bounded on X_∞, but not on X_0 and X_1.