Instructions: Do ***five (5)*** *problems, at least two from Part A, at least two from Part B, and one other from either part.* Start each chosen problem on a fresh sheet of paper and write your name at the top of each sheet.

Part A

1. Let H be a proper normal subgroup of a group G, and let K be a subgroup of H.
   (a) Give an example of this situation where K is not a normal subgroup of G.
   (b) Prove that if the normal subgroup H is cyclic, then K is normal in G.
   (c) Give a proof or a counterexample for the following claim:

   If H and G/H are both cyclic, then G is abelian.

2. (a) List (up to isomorphism) all abelian groups of order 32, as direct sums of cyclic groups.
   (b) Let H be the subgroup of $\mathbb{Z}^3$ generated by $\{(12, 10, 8), (0, 2, 0), (10, 8, 8)\}$ and let $G = \mathbb{Z}^3/H$. Show that G has order 32. To which of the groups listed in (a) is it isomorphic?

3. Let $\alpha := (1\,2)(3\,4\,5) \in S_5$.
   (a) Find a non-identity element of S_5 other than α that is conjugate to α. Demonstrate that it is conjugate.
   (b) Find an element of S_5 other than the identity that is *not* conjugate to α and demonstrate that is is not.
   (c) Does S_5 contain an element of order 6 that is not conjugate to α? Justify your answer.

4. Suppose a group G of order 77 acts on a set X of cardinality 24. Show that there are at least two different $x \in X$ that satisfy: $gx = x$ for all $g \in G$.

Part B

1. (a) Show that a finite integral domain is a field.
   (b) Give an example of an integral domain with exactly 4 elements.

2. Consider the ring $\mathbb{Z}[X]$ of polynomials in one variable X with coefficients in $\mathbb{Z}$.
   (a) Find all the units of $\mathbb{Z}[X]$.
   (b) Describe an easy way to recognize the elements of the ideal I of $\mathbb{Z}[X]$ generated by 2 and X.
   (c) Find a maximal ideal of $\mathbb{Z}[X]$.
   (d) Find a prime ideal of $\mathbb{Z}[X]$ that is not maximal.

3. Let R be a ring and M an R-module.

   (a) What does it mean for M to be a *free* R-module?

   (b) Let $\mathbb{Z}\left[\frac{1}{2}\right]$ denote the sub**ring** of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{2}$. Prove or disprove: $\mathbb{Z}\left[\frac{1}{2}\right]$ is a free $\mathbb{Z}$-module.