Instructions: Do ***five*** *of the 7 problems, including at least one from Part A, one from Part B, and one from Part C.* The remaining two problems can be from any parts. Start each chosen problem on a fresh sheet of paper and write your name at the top of each sheet. Clip your papers together in numerical order of the problems chosen when finished. You have three hours. Good luck!

Part A

1. (a) Let G be a group of order pq where $p < q$ are prime numbers. If G admits a normal subgroup of order p, prove that G is a cyclic group.

   (b) Give an example of a non-abelian finite group of order pq, where p, q are distinct primes.

2. Let $G = \mathrm{GL}(2, \mathbb{F}_p)$ be the group of invertible 2×2 matrices with entries in the finite field $\mathbb{F}_p$, where p is a prime.

   (a) Show that G has order $(p^2 - 1)(p^2 - p)$.

   (b) Determine the center of G and its isomorphism class. Please explain your answer.

Part B

3. Let $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$.

   (a) Show that $R/\langle 1 + \sqrt{-5}\rangle \cong \mathbb{Z}/6\mathbb{Z}$ as rings.

   (b) Is $\langle 1 + \sqrt{-5}\rangle$ a prime ideal of R? Give a reason for your answer.

4. Express the polynomial $X^4 + X^3 + X^2 + X + 1$ as a product of irreducible polynomials over each of the following fields: $\mathbb{Q}, \mathbb{C}, \mathbb{F}_5$ and $\mathbb{F}_2$.

5. (a) Define prime ideal and maximal ideal in a commutative ring R with identity.

   (b) Let R and S be commutative rings with identities 1_R and 1_S, respectively and $f : R \to S$ a ring homomorphism such that $f(1_R) = 1_S$. If P is a prime ideal of S, show that $f^{-1}(P)$ is a prime ideal of R.

   (c) Let f be as in part (b). If M is a maximal ideal of S, is $f^{-1}(M)$ a maximal ideal of R? Prove that it is or give a counterexample.

Part C

6. Let N be the subgroup of $\mathbb{Z}^3$ generated by $(4, 6, 4), (12, 12, 8)$ and $(10, 6, 8)$.

   (a) Determine the invariant factors and the elementary divisors of N. Please explain your answer.

   (b) Determine the invariant factor decomposition and the elementary divisors decomposition of the quotient group $\mathbb{Z}^3/N$. Please explain your answer.

7. Let R be a ring with identity and let $f : M \to N$ be a *surjective* homomorphism of R-modules, where N is a free R-module.

   (a) Show that there exists an R-module homomorphism $g : N \to M$ such that $f \circ g = \mathrm{id}_N$.

   (b) Show that $M \cong \ker(f) \oplus N$ as R-modules.