Instructions: Do ***five*** *of the 8 problems, including at least one from Part A, one from Part B, and one from Part C.* The remaining two problems can be from any parts. Start each chosen problem on a fresh sheet of paper and write your name at the top of each sheet. Clip your papers together in numerical order of the problems chosen when finished. You have three hours. Good luck!

Part A

1. (a) Decompose the following permutation into disjoint cycles:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 5 & 8 & 2 & 6 & 4 & 7 \end{pmatrix}$$

   (b) How many conjugacy classes does the symmetric group S_6 have? Explain your answer.

   (c) How many Sylow 3-subgroups H does S_6 have? What is the isomorphism type of H?

2. (a) Let A_4 be the alternating group of 4 elements. Show that A_4 has a normal subgroup of index 3.

   (b) Let $G = \mathrm{SL}_2(\mathbb{F}_2)$ be the group of 2 by 2 matrices with entries in the finite field $\mathbb{F}_2 = \mathbb{Z}/2$ which have determinant 1 (modulo 2). How many elements does this group have?

   (c) Construct a nontrivial homomorphism $G \to S_3$. Is it an isomorphism?

3. (a) Let G be a group with 21 elements. Show that G has a normal subgroup with 7 elements. Hint: Sylow theorem.

   (b) If G is abelian, show that it is isomorphic to a product of cyclic groups $C_3 \times C_7$.

   (c) Is there a nonabelian group with 21 elements? Can you describe one?

Part B

4. Let $R = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$.

   (a) Show that $1 + \sqrt{2}$ is a unit of R.

   (b) Determine whether $3 + \sqrt{2}$ is a prime element of R. Prove your assertion.

   (c) Identity the isomorphism class of the quotient ring $R/\langle 3 + \sqrt{2}\rangle$. Prove your assertion.

5. (a) Show that the polynomial $f(X) = X^3 - 25X^2 + 10X - 15$ is irreducible in $\mathbb{Q}[X]$.

   (b) Is it irreducible in $\mathbb{Z}[X]$? Explain.

   (c) Show that $\mathbb{F}_7[X]/\langle X^3 - 25X^2 + 10X - 15\rangle$ is a field with 343 elements.

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6. (a) Let $f : A \to B$ be a homomorphism of commutative rings with identity, $f(1) = 1$. If $J \subset B$ is an ideal, then show that $f^{-1}J \subset A$ is an ideal.

   (b) Show that f induces an injective ring homomorphism $\bar{f} : A/f^{-1}J \to B/J$.

   (c) Define what it means for an ideal $P \subset B$ to be a *prime* ideal. Then show that if P is a prime ideal, then $f^{-1}P \subset A$ is a prime ideal.

Part C

7. Let G be the abelian group with generators x, y, and z subject to the relations

$$\begin{array}{rcrcrcl} 3x & + & 6y & - & 3z & = & 0\,, \\ -3x & - & 4y & + & 3z & = & 0\,, \\ 6x & + & 12y & & & = & 0\,. \end{array}$$

   Determine the invariant factors and elementary divisors of G and write G as a direct sum of cyclic groups. Show the work for your assertions.

8. Let $f(X) = (X^2 + 5)(X - 2) \in \mathbb{Q}[X]$.

   (a) Construct all the matrices A in $M_5(\mathbb{Q})$, up to similarity, with minimal polynomial $m_A(X) = f(X)$ and determine their corresponding characteristic polynomials $\chi_A(X)$.

   (b) Construct all the Jordan canonical forms A in $M_5(\mathbb{C})$, up to similarity, with minimal polynomial $m_A(X) = f(X)$.