Mathematics Comprehensive Examination
Core-1 Algebra
January 2004

Directions: Do the first three problems and choose two problems from the last three problems, for a total of FIVE out of the six problems below.

Start each problem on a new sheet of paper. Please put your name and the problem number at the top of every sheet. Hand in ONLY five problems.

You have two and a half hours. GOOD LUCK!

1. a. Find the invariant factors of the group $\mathbb{Z}_{30} \oplus \mathbb{Z}_{24} \oplus \mathbb{Z}_{10}$.

   b. Find the number of nonisomorphic abelian groups of order $2^5 \cdot 3^2 \cdot 5^2$ that have no element of order eight.

2. a. Produce the lattice of subgroups of the alternating group A_4.

   b. Find five subgroups of order four of the alternating group A_5.

3. Show that the rings $\mathbb{F}_2[X]/(X^3 + X + 1)$ and $\mathbb{F}_2[X]/(X^3 + X^2 + 1)$ are fields of order eight. Verify that their multiplicative groups are cyclic of order seven.

4. a. Define what prime ideals and maximal ideals are. Give an example of an integral domain R and a prime ideal that is not maximal in R.

   b. Define what prime elements and irreducible elements are. Give an example of an integral domain R and an irreducible element that is not prime in R.

5. Consider the $\mathbb{Z}$-module A, given by the additive group of the field $\mathbb{Q}$ of rational numbers.

   Prove: A is neither a finitely generated $\mathbb{Z}$-module nor a free $\mathbb{Z}$-module.

6. Let $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$.

   Show that M is not diagonalizable over the finite field $\mathbb{F}_2$. Is M diagonalizable over the field $\mathbb{Q}$ of rational numbers?