ALGEBRA QUALIFYING EXAM

January 2023

Answer five of the following ten questions, including at least one from each of parts I, II, and III.

Part I

(1) (a) State the structure theorem for finitely generated abelian groups.
  (b) If p and q are distinct primes, determine the number of nonisomorphic abelian groups of order p^3q^4.

(2) Prove that every finitely generated subgroup of the additive group of rational numbers is cyclic.

(3) Let G be a finite abelian group of odd order. If $\varphi : G \to G$ is defined by $\varphi(a) = a^2$, show that φ is an isomorphism. Generalize this result.

(4) Prove that a group of order 30 can have at most 7 subgroups of order 5.

Part II

(5) Let R be an integral domain. Show that the group of units of the polynomial ring $R[X]$ is equal to the group of units of the ring R.

(6) Let $\omega = (1 + \sqrt{-3})/2 \in \mathbb{C}$, and let $R = \{a + b\omega : a, b \in \mathbb{Z}\}$.
  (a) Show that R is a subring of $\mathbb{C}$.
  (b) Show that R is a Euclidean domain with respect to the norm function $N(z) = z\bar{z}$, where, as usual, $\bar{z}$ denotes the complex conjugate of z.

(7) Let R be a commutative ring with 1. Recall that an element $r \in R$ is *nilpotent* if there is an integer $n \geq 1$ such that $r^n = 0$. Let $\mathfrak{N}(R)$ be the set of nilpotent elements in R. Show that $\mathfrak{N}(R)$ is an ideal in R.

Part III

(8) (a) Show that $\mathbb{Q}$ is a torsion-free $\mathbb{Z}$-module.
  (b) Is $\mathbb{Q}$ a free $\mathbb{Z}$-module? Justify your answer.

(9) Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$, where d is the greatest common divisor of n and m.

(10) Let A be an $n \times n$ matrix with entries in a field F. Show that if $A^2 = A$, then A is similar to a diagonal matrix which has only 0's and 1's on the diagonal.