ALGEBRA QUALIFYING EXAM

January 2024

Answer five of the following ten questions, including at least one from each of parts I, II, and III.

Part I

(1) Let $G = \mathrm{GL}(2, \mathbb{F}_p)$ be the group of invertible 2×2 matrices with entries in the finite field $\mathbb{F}_p$, where p is a prime. Show that G has order $(p^2-1)(p^2-p)$.

(2) Let G be a group of order $2p$ where p is an odd prime. If G has a normal subgroup of order 2, show that G is cyclic.

(3) Prove that the product of two infinite cyclic groups is not cyclic.

(4) Let G be a group of order 132. Show that G is not simple.

Part II

(5) (a) Give an example of an integral domain with exactly 9 elements.
(b) Is there an integral domain with exactly 10 elements? Justify your answer.

(6) Let R be an integral domain. Show that the group of units of the polynomial ring $R[x]$ is equal to the group of units of the ring R.

(7) Let R be a PID. Prove that every nonzero prime ideal in R is a maximal ideal.

Part III

(8) Let $\mathbb{Z}[\frac{1}{2}]$ be the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{2}$. Prove or disprove: $\mathbb{Z}[\frac{1}{2}]$ is a free $\mathbb{Z}$-module.

(9) Let $M \subset \mathbb{Z}^n$ be a $\mathbb{Z}$-submodule of rank n. Prove that $\mathbb{Z}^n/M$ is a finite group.

(10) Suppose that A is a 3×3 complex matrix such that

$$A^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Show that A is diagonalizable.