Directions. *Do problems 1, 2, and 3, plus any two of the remaining four problems. Start each problem on a new sheet of paper, and put your name and the problem number at the top of every sheet. Hand in* only *the five problem that you want graded. The time available for the exam is two and one-half hours.* Good luck!

---

1. Let G be a group of order $2p$ where p is an odd prime. If G has a normal subgroup of order 2, show that G is cyclic.

2. (a) Define *prime ideal* and *maximal ideal* in a commutative ring R with identity.

   (b) Let R be a commutative ring with identity. If M is a maximal ideal of R, prove that M is a prime ideal.

   (c) Give an example of a ring R and a nonzero prime ideal P of R such that P is *not* maximal. Naturally, you are expected to show that your example has the requisite properties.

3. Find the characteristic polynomial, minimal polynomial, and Jordan canonical form of the linear transformation T with matrix

$$\begin{bmatrix} 4 & 0 & 4 \\ 2 & 1 & 3 \\ -1 & 0 & 0 \end{bmatrix}.$$

---

4. Let $G = (\mathbb{Z}/247\mathbb{Z})^*$ be the group of units of the ring $\mathbb{Z}/247\mathbb{Z}$. (Note that $247 = 13 \cdot 19$.)

   (a) Determine the order of G.

   (b) Determine the invariant factor decomposition of G.

   (c) Determine the elementary divisor decomposition of G.

5. (a) Show that $\mathbb{Z}[i]/\langle 3+i \rangle \cong \mathbb{Z}/10\mathbb{Z}$, where i is the usual complex number $\sqrt{-1}$.

   (b) Is $\langle 3+i \rangle$ a maximal ideal of $\mathbb{Z}[i]$? Give a reason for your answer.

6. (a) Let R be a ring and M an R-module. What does it mean for M to be a *free* R-module? In other words, give a *precise* definition.

   (b) Let $\mathbb{Z}\left[\frac{1}{2}\right]$ denote the *subring* of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{2}$. Prove or disprove: $\mathbb{Z}\left[\frac{1}{2}\right]$ is a free $\mathbb{Z}$-module.

7. Let p be a prime number and let V be a 2-dimensional vector space over the field $\mathbb{F}_p$ with p elements.

   (a) What is the number of elements in V?

   (b) Find the number of linear transformations $T : V \to V$.

   (c) Find the number of invertible linear transformations $T : V \to V$.