LSU Mathematics Comprehensive Examination
Core-1 Algebra Exam
January 2003

Directions: Do at least one problem from the set {G1, G2} (on groups); at least one problem from the set {R1, R2, R3} (on rings); at least one problem from the set {M1, M2} (on modules); and at least one problem from the set {L1, L2, L3} (on linear algebra). Then do two more problems from anywhere on the list, for a total of **six**. Please start each problem on a new sheet of paper, with your name and the problem number written at the top of every sheet. Hand in only six problems. You have two and a half hours. Enjoy.

Here, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ denote the rings of integers, rational numbers, and complex numbers, respectively.

G1. (a) List all abelian groups of order 2700 (up to isomorphism). Brief justification.

(b) Give the elementary divisors and invariant factors of the group $Z_{75} \oplus Z_{12} \oplus Z_3$.

G2. The first column of the following table lists the possible cycle structures for permutations in S_5. (For example, the cycle structure of $(3\,5)(4\,1\,2)$ is $(1\,2)(3\,4\,5)$.) Complete the table by filling in the columns for the number of permutations in S_5 of each cycle structure, the order of each such permutation, and the parity (i.e., even or odd). (Note that one of the requested numbers has already been given, to get you started.) What is the order of S_5?

Cycle Structure	Number	Order	Parity
(1)			
$(1\,2)$			
$(1\,2\,3)$	$\frac{5\cdot 4\cdot 3}{3}$		
$(1\,2\,3\,4)$			
$(1\,2\,3\,4\,5)$			
$(1\,2)(3\,4)$			
$(1\,2)(3\,4\,5)$			

R1. Let R be a commutative ring with identity. Let a be a unit in R, and let b be a nilpotent element of R. Show that $a + b$ is a unit in R.

R2. Let $m, n \in \mathbb{Z}$ be relatively prime and positive.

(a) Let F be a field. Let $a, b \in F$ satisfy $a^n = b^n$ and $a^m = b^m$. Prove that $a = b$.

(b) Does the conclusion of (a) above remain true when the field F is replaced by the integral domain $\mathbb{Z}$? Explain.

R3. In (a) and (b) below, let R be an integral domain, and let $a \in R$.

(a) What does it mean for a to be *prime*? What does it mean for a to be *irreducible*?

(b) Show that if a is prime, then it is irreducible.

(c) Give an example of an integral domain R, and an irreducible element $a \in R$ that is not prime.

M1. Let R be a principal ideal domain, and let $p, q \in R$ be relatively prime.
Let M be an R-module, and let $x, y \in M$. Write $\mathcal{O}_x$ and $\mathcal{O}_y \subseteq R$ for the order-ideals of x and y, respectively. (Note: Some authors write these ideals as $\operatorname{Ann}(x)$ and $\operatorname{Ann}(y)$.) Suppose $\mathcal{O}_x = (p)$ and $\mathcal{O}_y = (q)$. Show that $\mathcal{O}_{x+y} = (pq)$.

M2. Let R be a ring. Let A, B be left R-modules.

(a) Let $f : A \to B$ be an R-module homomorphism. Prove that $\operatorname{Ker} f$ is a submodule of A. Then prove the First Isomorphism Theorem for modules, viz., that f induces an R-module isomorphism $\overline{f} : A/(\operatorname{Ker} f) \to \operatorname{Im} f$.

(b) Let $x \in A$. Show that there is an R-module isomorphism $R/\mathcal{O}_x \to Rx$.
(Here, as in M1 above, $\mathcal{O}_x$ denotes the order-ideal of x.)

L1. Let $V := \{\, f(X) \in \mathbb{C}[X] \mid \deg f \leq 3 \,\}$ be the $\mathbb{C}$-vector space of complex polynomials of degree ≤ 3 in X. Define $\phi : V \to V$ by $\phi(f(X)) := f(X+1) - f(X)$. For example,

$$\phi(X^2 + X + 2) = ((X+1)^2 + (X+1) + 2) - (X^2 + X + 2) = 2X + 2.$$

Let $\mathcal{B} = \{1,\ X,\ X(X-1),\ X(X-1)(X-2)\}$.

(a) Show that $\mathcal{B}$ is a basis of V.

(b) Find the matrix of ϕ relative to $\mathcal{B}$.

L2. Let V be a vector space over $\mathbb{C}$, and let $\phi : V \to V$ be a linear transformation with invariant factors $q_1 := (X^2+1)(X-3)$ and $q_2 := (X^2+1)^2(X-3)^3$. Write the Jordan canonical form of ϕ (relative to a suitable basis, which you need not show). Also write the characteristic polynomial and the minimal polynomial of ϕ.

L3. Let F be a field, let $V_1, \ldots, V_{n+1}$ be vector spaces over F, and let

$$0 \to V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_n} V_{n+1} \to 0$$

be a sequence of linear transformations that is *exact* (this means that ϕ_1 is injective, ϕ_n is surjective, and $\operatorname{Im} \phi_i = \operatorname{Ker} \phi_{i+1}$, for all $i \in \{1, \ldots, n-1\}$). Show that

$$\sum_{i=1}^{n+1} (-1)^i \dim V_i = 0.$$