Algebra I Comprehensive Exam

January 10, 2008

Instructions: Do any **five (5)** of the following **six (6)** problems. *Start each chosen problem on a fresh sheet of paper.* **Write your name on each sheet at the top**. Be sure to cite all theorems that you apply, checking that all hypotheses are satisfied. You have 3 hours for this test. Good luck!

1. Let G be a group. and let Z denote the center of G.
   (a) Show that Z is a normal subgroup of G.
   (b) Show that if G/Z is cyclic, then G must be abelian.
   (c) Let D_6 be the dihedral group of order 6. Find the center of D_6.
2. Let R be a ring with 1. An R-module N is called *simple* if it is not the zero module and if it has no submodules except N and the zero submodule.
   (a) Prove that any simple module N is isomorphic to R/M, where M is a maximal ideal.
   (b) Prove *Schur's Lemma:* Let $\varphi : S \to S'$ be a homomorphism of simple modules. Then either φ is zero, or it is an isomorphism.
3. (a) Show that no group of order 30 is simple.
   (b) Are there nonabelian groups of order 30? Prove your answer.
4. Let $R = \mathbf{Z}[X]$. Answer the following questions about the ring R. You may quote an appropriate theorem, provide a counterexample, or give a short proof to justify your answer.
   (a) Is R a unique factorization domain?
   (b) Is R a principal ideal domain?
   (c) Find the group of units of R.
   (d) Find a prime ideal of R which is not maximal.
   (e) Find a maximal ideal of R.
5. Let R be a commutative ring with 1. Show that an R-module M is Noetherian if and only if every submodule of M is finitely generated.
6. Let F be a field. Construct, up to similarity, all linear transformations $T : F^6 \to F^6$ with minimal polynomial $m_T(X) = (X-5)^2(X-6)^2$. In each case, give the characteristic polynomial.