Instructions: Do five of the six problems. Turn in only the five problems you want graded. Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. Start each problem on a new sheet of paper. You have three hours. Good luck!

---

1. (a) State the definition of conjugacy class of an element of a group.

   (b) Find the centralizer of the cycle (1234) in the symmetric group S_4.

   (c) Find the cardinality of the conjugacy class of the cycle $(1234) \in S_4$.

2. Assume that $T\colon \mathbb{R}^5 \to \mathbb{R}^5$ is a linear transformation which satisfies $T^3 = T^2$. Find all choices for T up to similarity.

3. Let M be an R-module and let $f\colon M \to M$ be an R-module endomorphism which is idempotent, that is $f \circ f = f$. Prove that $M \cong \mathrm{Ker} f \oplus \mathrm{Im} f$.

4. State and prove Gauss' lemma for primitive polynomials over a ring which is a unique factorization domain.

5. Let G be a group and H be a subgroup of G.

   (a) Show that G acts on the set of left cosets for H in G by
   $$g.(g'H) = gg'H, \quad g, g' \in G.$$

   (b) Assume that the index of H in G is finite. Use part (a) to show that G contains a normal subgroup K such that $K \leq H$ and $[G : K]$ is finite.

6. (a) Prove that the ring
   $$\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$$
   is a Euclidean domain.

   (b) Compute the norm of the element $3 + 5\sqrt{-2}$ and show that it is a prime element of $\mathbb{Z}[\sqrt{-2}]$.