Instructions: Complete any **five (5)** of the following problems. Turn in only these five problems to be graded. Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours. Good luck!

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1. (a) For $n \geq 1$, define the alternating group A_n as a subgroup of the symmetric group S_n. What are the orders of A_n and S_n?

   (b) Prove that for $n \geq 3$, A_n contains a subgroup isomorphic to S_{n-2}.

2. Prove that if a group has exactly one element of order two, then that element is in the center of the group.

3. (a) Prove that $R = \mathbb{Z}[\sqrt{3}]$ is an integral domain.

   (b) Prove that $2 + \sqrt{3}$ is a unit in R.

   (c) Prove that $F = \mathbb{Q}[\sqrt{3}]$ is a field.

4. (a) Prove that a Euclidean domain is a Principal Ideal Domain.

   (b) Give an example of a Unique Factorization Domain that is not a Principal Ideal Domain. Justify your answer.

5. Suppose that R is an integral domain, M is an R-module, and $N \subset M$ is an R-submodule. For each of the following statements, determine whether it is true or false, and provide a proof or counterexample as appropriate:

   (a) If M is free, then N is free.

   (b) If M is torsion-free, then N is torsion-free.

   (c) If M is cyclic, then N is cyclic.

6. (a) State the Fundamental Theorem of Finitely Generated Modules over Principal Ideal Domains.

   (b) Let $G := \mathbb{Z}^2$ and let H be the subgroup generated by (m, n), where m and n are integers. Determine the structure of the abelian group G/H; i.e., find the free rank and invariant factors.