Instructions: Do ***five*** *of the 10 problems, including at least one from Part A, one from Part B, and one from Part C.* The remaining two problems can be from any parts. Start each chosen problem on a fresh sheet of paper and write your name at the top of each sheet. Clip your papers together in numerical order of the problems chosen when finished. You have three hours. Good luck!

Part A

1. (a) Are (123)(45) and (235)(14) conjugate in $|S_5|$? If so, find an element giving the conjugation.

   (b) Identify the isomorphism class of each Sylow p-subgroup of S_5. Please prove your assertion.

2. Let G be the group of invertible upper triangular 2×2 matrices with entries in $\mathbb{R}$. Let $D \subset G$ be the subgroup of diagonal matrices, and let $U \subset G$ be the subgroup of matrices of the form $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ with $x \in \mathbb{R}$.

   (a) Show that the group U is isomorphic to the additive group $\mathbb{R}$.

   (b) Show that U is a normal subgroup of G and that G/U is isomorphic to D.

   (c) Is G isomorphic to $U \times D$? Justify your answer.

3. Let D_8 be the group of symmetries of a square.

   (a) Give a presentation of D_8 in terms of generators and relations.

   (b) Give an injective group homomorphism $D_8 \to S_4$.

Part B

4. Let $\mathbb{F}_5 = \mathbb{Z}/5$ be the field with 5 elements.

   (a) Show that the polynomials $f(x) = x^2+2$ and $g(x) = x^2+x+1$ in $\mathbb{F}_5[x]$ are irreducible.

   (b) Show that $K = \mathbb{F}_5[\alpha] = \mathbb{F}_5[x]/\langle f(x)\rangle$ and $L = \mathbb{F}_5[\beta] = \mathbb{F}_5[x]/\langle g(x)\rangle$ are fields. Both K and L have the same number of elements. How many?

   (c) Give an explicit isomorphism $K \cong L$.

5. Let R be a commutative ring with identity, and I, J ideals of R.

   (a) If $I + J = R$, prove that $IJ = I \cap J$.

   (b) If R is a PID and $IJ = I \cap J$, prove that $I + J = R$.

   (c) Show that $\mathbb{Z}/60\mathbb{Z}$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ as rings.

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6. Let $R = \mathbb{Z}[x]$.
   (a) Show that $M = \langle 3, x \rangle \subset R$ is a maximal ideal. What is R/M?
   (b) Show that $P = \langle 5x^2 + 3x - 1 \rangle \subset R$ is a prime ideal. Is it maximal?
   (c) Is $\langle 5x^2 + 3x - 1 \rangle \subset S$ a maximal ideal in the ring $S = \mathbb{Q}[x]$? Explain.

Part C

7. (a) Reduce the matrix $A = \begin{bmatrix} 4 & 7 & 2 \\ 2 & 4 & 6 \end{bmatrix}$ to diagonal form (Smith normal form) by integer row and column operations. $D = P^{-1}AQ$.
   (b) If we think of $A : \mathbb{Z}^3 \to \mathbb{Z}^2$, find a basis for $\mathrm{Ker}(A)$.
   (c) What is the structure of the quotient $\mathbb{Z}^2/A\mathbb{Z}^3$?

8. (a) Give an example of a split short exact sequence of abelian groups.
   (b) Give an example of a short exact sequence of abelian groups which is not split.
   (c) If
   $$M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \longrightarrow 0$$
   is an exact sequence of abelian groups, show that there is an exact sequence
   $$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M_3, N) \xrightarrow{\beta^*} \mathrm{Hom}_{\mathbb{Z}}(M_2, N) \xrightarrow{\alpha^*} \mathrm{Hom}_{\mathbb{Z}}(M_1, N)$$
   for every abelian group N.

9. Let $f(X) = (X^2 + 1)(X - 2)^2 \in \mathbb{Q}[X]$.
   (a) Construct all the matrices A in $M_6(\mathbb{Q})$, up to similarity, with minimal polynomial $m_A(X) = f(X)$ and determine their corresponding characteristic polynomials $\chi_A(X)$.
   (b) Construct all the Jordan canonical forms A in $M_6(\mathbb{C})$, up to similarity, with minimal polynomial $m_A(X) = f(X)$.

10. Let $A \in M_3(\mathbb{Q})$ be the matrix
   $$\begin{bmatrix} -1 & -2 & 6 \\ -1 & 0 & 3 \\ -1 & -1 & 4 \end{bmatrix}.$$
   (a) Find the eigenvalues and eigenvectors of the matrix A.
   (b) What is the Jordan canonical form of A?

(c) The Smith normal form of $xI - A$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & & x^2-2x+1 \end{bmatrix}.$$

If the column space $V = \mathbb{Q}^3$ is a $\mathbb{Q}[x]$-module by the rule $x \cdot v = Av$, what is the structure of the $\mathbb{Q}[x]$-module V? That is, express it as a sum of cyclic modules (what are the invariant factors)?