Instructions: Do ***five (5)*** *of the 8 problems, including* ***at least two (2) from Part A*** and ***at least two (2)*** *from* ***Part B***. Each of your 5 problems counts for 20 points. *Start each chosen problem on a fresh sheet of paper.* **Write your name at the top of each sheet. Please turn in 5 problems, even if the solutions are imperfect, for partial credit.** Logical rigor is important. Describe which theorems you apply, *showing how you check that all hypotheses are satisfied*. If you see a gap in your proof but not how to fix it, state this because it is better that you recognized it! *Clip your papers together in numerical order of the problems chosen when finished.* You have 3 hours. **Good luck!**

Symbols: Integration is in the sense of Lebesgue Measure and Integration theory. Lebesgue measure on $\mathbb{R}$ or $\mathbb{R}^n$ is l, and $(X, \mathfrak{A}, \mu)$ is an abstract measure space. Also, 1_S is the *characteristic*, or *indicator*, function of a set S, and $A \triangle B = (A \cup B) \setminus (A \cap B)$, the symmetric difference of A and B.

Part A: Measures

1. Let $(X, \mathfrak{A}, \mu)$ be a measure space for which $\mu(X) < \infty$.

- **a.** (10) Suppose f_n is a sequence of measurable functions such that $f_n \to f$ pointwise everywhere. Prove that $f_n \to f$ in measure, meaning that for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \implies \mu\left\{x \,\middle|\, |f_n(x) - f(x)| > \epsilon\right\} < \epsilon$.
- **b.** (5) Give an example of $f_n \to f$ pointwise everywhere on $[0, 1]$ yet $l\left\{x \,\middle|\, |f_n(x) - f(x)| > \epsilon\right\} > 0$ for all $\epsilon > 0$.
- **c.** (5) Give an example on $[0, 1]$ of a sequence of measurable functions f_n converging in measure, yet $f_n(x)$ fails to converge for any x.

2. Let $E \subset \mathbb{R}$ be a measurable set with the property that $l(E \cap I) \leq \dfrac{l(I)}{2}$ for every open interval I. Prove that $l(E) = 0$. (Hints: Show it suffices to prove this if $l(E) < \infty$. Then note that if E is measurable then the outer measure $l^*(E) = l(E)$.)

3. Suppose A and B are measurable subsets of $\mathbb{R}$, each one of strictly positive but finite measure. Prove that there exists a number $c \in \mathbb{R}$ such that $l((A + c) \cap B) > 0$. (Hints: One method is to consider the *outer measures* $l^*(A)$ and $l^*(B)$ in order to give a direct measure theoretic proof. An *easier alternative solution* is to consider the convolution $1_{-A} * 1_B(x) = \int_{\mathbb{R}} 1_{-A}(x - y) 1_B(y)\, dy$ and use Fubini's theorem with justification.)

4. Let $f \geq 0$ be a Lebesgue measurable function on $\mathbb{R}^d$. Define $\mu(E) = \int_E f\, dl$.

- **a.** (5) Show that μ is absolutely continuous with respect to Lebesgue measure: $\mu \prec l$.
- **b.** (5) Show that μ is *purely infinite* on the set $E = \{x \mid f(x) = \infty\}$. (Definition: μ is purely infinite on E if and only if $\mu(F) = 0$ or $\mu(F) = \infty$ on any measurable subset F of E.)
- **c.** (10) Show that μ is σ-finite on the set $E = \{x \mid f(x) < \infty\}$. Note this means there is a sequence E_n of measurable sets with $E = \bigcup_{n=1}^{\infty} E_n$ and $\mu(E_n) < \infty$ for all n.

Part B: Integrals

5. Let $f(x) = x^{-r}$ where $r < 1$.

- **a.** Show that $f \in L^1[0,1]$.
- **b.** Let $a_n = \displaystyle\int_0^1 \frac{1}{\frac{1}{n} + x^r}\, dx$. Compute $\lim_{n\to\infty} a_n$. Use appropriate integration theorems to justify your calculations carefully.

6. Suppose that $f \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} |xf(x)|\, dl(x) < \infty$. Define the Fourier sine transform F of f by

$$F(\alpha) = \int_{\mathbb{R}} f(x) \sin(\alpha x)\, dl(x)$$

for all $\alpha \in \mathbb{R}$. Prove that the *derivative* $F'(\alpha) = \dfrac{d}{d\alpha} \displaystyle\int_{\mathbb{R}} f(x) \sin(\alpha x)\, dl(x)$ *exists* and find its value for all $\alpha \in \mathbb{R}$. *Justify carefully* bringing the differentiation with respect to α inside the integral.

7. Prove that if f_n is Lebesgue integrable on $[0,1]$ for each $n \in \mathbb{N}$ and $\displaystyle\sum_{n\in\mathbb{N}} \int_0^1 |f_n(x)|\, dx < \infty$, then $\sum_{n\in\mathbb{N}} f_n(x)$ is convergent almost everywhere, and $\displaystyle\int_0^1 \sum_{n\in\mathbb{N}} f_n(x)\, dx = \sum_{n\in\mathbb{N}} \int_0^1 f_n(x) dx$. (Hint: Interpret the summation as an integral over $\mathbb{N}$ with respect to counting measure ν. Justify the use of each theorem you apply.)

8. Prove: If $f \in L^1(\mathbb{R})$, then $\displaystyle\lim_{x\to 0} \int_{\mathbb{R}} |f(x+t) - f(t)|\, dl(t) = 0$. Suggested steps:

- **a.** Let $f = \sum_{i=1}^{K} c_i 1_{[a_i,b_i]}$, an integrable step function. Let $M = \|f\|_\infty$. The set of all x such that *exactly one* of $x, x+t$ lies in one of the intervals $[a_i, b_i]$ is

$$E = \bigcup_{1\le i\le K} \big([a_i, b_i] \triangle [a_i - t, b_i - t]\big).$$

  Bound the measure $l(E)$ from above. Show that $\|f_t - f\|_1 \to 0$ as $t \to 0$ where $f_t(x) = f(x+t)$.

- **b.** Prove the claim for general $f \in L^1(\mathbb{R})$. (Hint: Use a result about dense subsets of $L^1(\mathbb{R})$.)