Solve one of the problems (1)–(2), two of the problems (3)–(6) and two of the problems (7)–(9). Only turn in the solution to at most **five** problems.

In the following λ denotes the Lebesgue measure on one of the spaces $\mathbb{R}^n$ and λ_k denotes the Lebesgue measure on $\mathbb{R}^k$. If not otherwise stated, the statement that μ is a measure means that μ is σ-additive. The notation $\overline{\mathbb{R}}$ stands for the extended real numbers. The indicator function of a set A is denoted by χ_A.

Turn in all your work even if you do not finish a problem. You might get partial credit. Make sure that you have written you name on all pages that you turn in.

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1. Let $(X, \mathcal{A})$ be a measurable space and let $\mu : \mathcal{A} \to [0, \infty]$ be a **finitely additive measure**.
   (a) Show that if $\mu(X) < \infty$ then μ is countably additive if and only if for every decreasing sequence $\{A_j\}_{j\in\mathbb{N}}$ in $\mathcal{A}$ we have
   $$\mu\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{j\to\infty} \mu(A_j)\,. \tag{1}$$
   (b) Give an example of a measure space $(X, \mathcal{A}, \mu)$, (μ σ-additive) with $\mu(X) = \infty$, and a sequence of decreasing measurable sets $\{A_j\}_{j\in\mathbb{N}}$ in $\mathcal{A}$ such (1) fails.
2. Let $(X, \mathcal{A}, \mu)$ be a measure space such that $\mu(X) > 0$. Let $f : X \to \overline{\mathbb{R}}$ be measurable and suppose that f is finite μ almost everywhere. Show that there exists $Y \in \mathcal{A}$ such that $\mu(Y) > 0$ and f is bounded on Y.

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3. Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions $f_n : X \to \mathbb{R}$ that converges almost everywhere to a measurable function $f : X \to \mathbb{R}$. Show that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to f in measure.
4. Determine if the following limits exists. If the limit exists, find the limit and justify your answer:
   (a) $\displaystyle\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n dx.$
   (b) $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}^+} \frac{\sin(x/n)}{1 + x^2}\, dx.$
   (c) $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}} f(x) g_n(x)\, d\lambda(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $g_n(x) = n\chi_{[-1/2,1/2]}(nx)$.

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5. For $n = 1, 2, \ldots$ define

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if} \quad |x| \le n, \\ 0 & \text{if} \quad |x| > n \end{cases} .$$

Find

$$\lim_{n\to\infty} f_n = f \text{ and } \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, d\lambda(x) .$$

Explain why your result does not contradict the Lebesgue dominated convergence theorem.

6. Let $(X, \mathcal{A}, \mu)$ be a measure space. Assume that $f_n \in L^1(X, \mu)$ for $n \in \mathbb{N}$ and that

$$\sum_{n=1}^{\infty} \int_X |f_n(x)|\, d\mu(x) < \infty .$$

Show that

a) The series $\sum_{n=1}^{\infty} f_n(x)$ converges almost everywhere to a measurable function f on X.

b) $f \in L^1(X, \mu)$ and

$$\int_X f(x)\, d\mu(x) = \sum_{n=1}^{\infty} \int_X f_n(x)\, d\mu(x) .$$

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7. Let $f, g \in L^1(\mathbb{R}, \lambda_1)$.

(a) Show that $H(x, y) = f(x)g(y-x)$ is integrable on $\mathbb{R}^2$ with respect to the Lebesgue measure λ_2.

(b) Show that $x \mapsto H(x, y)$ is integrable with respect to λ_1, that $G(y) = \int_{\mathbb{R}} H(x, y)\, d\lambda_1(x)$ is integrable on $\mathbb{R}$ with respect to λ_1, and that

$$\|G\|_1 \le \|f\|_1 \, \|g\|_1 .$$

8. For $-\infty < a < b < \infty$ let $I = [a, b]$ and let μ be a signed measure on I (with respect to the Borel σ-algebra). Let $f(x) = \mu([a, x])$. Then f is of bounded variation.

9. Let $Q = [0, 1]^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (\forall j = 1, \ldots, n) 0 \le x_j \le 1\}$. Let $f : Q \to \mathbb{R}$ be a continuous function. Show that the Lebesgue measure of the set $\{(x, f(x)) \mid x \in Q\} \subset \mathbb{R}^{n+1}$ is zero.