Mathematics Comprehensive Examination

Core I –Analysis

August 2015

Directions: There are two parts to the exam. Solutions to at least two problems must be turned in from each part. Solutions to 5 problems are required.

Part I.

1. Let $E \subset \mathbb{R}$ be Lebesgue measurable. Suppose $\forall$ c, s.t., $0 < c < 1$ and all intervals I that $m(E \cap I) \leq cm(I)$. Show $m(E) = 0$.

2. Show if f is a Lebesgue measurable real valued function and g a continuous function defined on $(-\infty, \infty)$ then $g \circ f$ is measurable.

3. Let g be monotone increasing and absolutely continuous on $[a, b]$ with $g(a) = c$ and $g(b) = d$. Let m denote Lebesgue measure and show for any open set $\mathcal{O} \subset [c, d]$
$$m(\mathcal{O}) = \int_{g^{-1}(\mathcal{O})} g'(x)\, dx.$$

Part II.

1. Let f be defined by $f(0) = 0$ and $f(x) = x \sin\left(\frac{1}{x}\right)$ for $x \neq 0$. Is f of bounded variation on $[-1, 1]$? If it is prove why otherwise demonstrate that it is not.

2. Give a proof or provide a counterexample for the following statement. Continuous functions defined on $[a, b]$ are Lebesgue measurable functions.

3. Let $f(x, t)$ be defined on $\{(x, t) : 0 \leq x \leq 1,\, 0 \leq t \leq 1\}$ and suppose: 1) f is a Lebesgue measurable function of x for each fixed value of t and f is continuous in t for each x and 2) there is an integrable function $g(x)$ such that $g(x) \geq |f(x, t)|$. Show $h(t) = \int f(x, t)\, dx$ is a continuous function of t.

4. Let g be a monotone increasing absolutely continuous function on $[a, b]$ with $g(a) = c$, $g(b) = d$ and let f be an integrable function on $[c, d]$. Let
$$F(y) = \int_c^y f(t) dt$$
and set $H(x) = F(g(x))$. Show H is absolutely continuous and that $F'(g(x))$ exists whenever H' and g' exist and $g'(x) \neq 0$. Therefore $H'(x) = F'(g(x))g'(x)$ almost everywhere except on the set where $g'(x) = 0$.