Real Analysis Comprehensive/Qualifying Exam August 2019

The exam has three parts. You must turn in **two** problems from each of parts I and II and **one** problem from part III. All problems have the same weight. Only five problems will be graded. Make sure to have your name clearly written on the solution sheets. You can use "well known theorem" from the lectures, but make sure you state what theorem you are using and make sure you clearly argue that the conditions in the theorem are satisfied. $(X, \mathcal{A})$ will always stand for a measurable state and μ will denote a measure on X. If not otherwise stated no further conditions on $X, \mathcal{A}$ or μ are made. If $A \subseteq X$ then χ_A denotes the indicator functions of the set A. λ will denote the Lebesgue measure on $\mathbb{R}$ or a given subset of $\mathbb{R}$. λ^d will denote the Lebesgue measure on $\mathbb{R}^d$ or a subset thereof. $\overline{\mathbb{R}}$ denotes the set of extended real numbers, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$. We set $\mathbb{R}_{\geq 0} = [0, \infty)$.The convolution of two measurable functions on $\mathbb{R}^d$ is defined by $f * g(x) = \int_{\mathbb{R}^d} f(y)g(x-y)\, d\lambda(y)$ whenever $y \mapsto f(y)g(x-y)$ is integrable. You are allowed to use that if $A \subset \mathbb{R}$ is an interval (a,b), $(a,b]$, $[a,b)$ or $[a,b]$ or an unbounded interval and f is continuous and integrable the the Riemann integral and the Lebesgue integral are the same.

PART I

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1) a) Let $\{E_j\}$ be a sequence of measurable sets. Show that if $\{E_j\}$ is monotone (either increasing or decreasing) then

$$\limsup E_j = \liminf E_j \, .$$

b) Give an example of a sequence of measurable sets $\{E_j\}$ such that $\limsup E_j \neq \liminf E_j$.

2) Show that if $E \subseteq [0,1]$ is measurable and there exists a constant $c > 0$ such that $\lambda(E \cap I) \geq c\lambda(I)$ for all open intervals $I \subset [0,1]$, then $\lambda(E) = 1$.

3) Let X be a set and $\mathcal{A}$ an **algebra** of subsets of X. Let μ be finitely additive measure on $\mathcal{A}$ such that $\mu(X) < \infty$. Show that μ is countable additive on $\mathcal{A}$ if and only if for each decreasing sequence

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq A_{n+1} \supseteq \cdots , \quad A_n \in \mathcal{A}$$

with $\displaystyle\bigcap_{n=1}^{\infty} A_n = \emptyset$ we have $\displaystyle\lim_{n\to\infty} \mu(A_n) = 0$.

4) Let $1 \leq p < \infty$. For $n \in \mathbb{N}$ let $f_n = n^{-1/p}\chi_{[0,n]}$. Prove the following statements:

a) f_n converges uniformly to the zero function.

b) f_n converges in measure to the zero function.

c) f_n does not converge in $L^p(\mathbb{R}, \lambda)$ to the zero function.

PART II

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5) Let $f_n : [0, \infty) \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{x}{n} e^{-x/n}$, $n \in \mathbb{N}$.

a) Determine $f(x) = \lim\limits_{n\to\infty} f_n(x)$. Show that the sequence $(f_n)_n$ converges uniformly to f on $[0, R]$ for all $R > 0$. Does the sequence converge uniformly to f on $\mathbb{R}_{\geq 0}$.

b) Show that

$$\lim_{n\to\infty} \int_{[0,R]} f_n(x) d\lambda(x) = \int_{[0,R]} f(x) d\lambda(x)$$

but that

$$\lim_{n\to\infty} \int_{\mathbb{R}_{\geq 0}} f_n(x) d\lambda(x) \neq \int_{\mathbb{R}_{\geq 0}} f(x) d\lambda(x).$$

6) Let f be a real valued measurable function on a finite measure space $(X, \mathcal{A}, \mu)$. Show that

$$\lim_{n\to\infty} \int_X \cos^{2n}(\pi f(x)) d\mu(x) = \mu(\{x \in X \mid f(x) \in \mathbf{Z}\}).$$

7) For each of the following problems, check whether the limit exists. If so, find its value. Give precise statements and references to what theorems/lemmas you are using.

a) $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}_{\geq 0}} \left(1 + \frac{x}{n}\right)^{-n} \sin(x/n) d\lambda(x)$.

b) $\displaystyle\lim_{n\to\infty} \int_{[0,n]} \frac{\sin(x)}{1 + nx^2} d\lambda(x)$.

8) a) Let $f \in \mathcal{L}^1(\mathbb{R}_{\geq 0}, \lambda)$. Show that $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}_{\geq 0}} e^{-nx} f(x) d\lambda(x)$ exists and find the limit.

b) Show that for all $n \in \mathbb{N}$ the function $f_n(x) = \dfrac{n}{1 + n^2x^2}$ is integrable on $[1, \infty)$. Then find $\lim_{n\to\infty} \int_{[1,\infty)} f_n(x) d\lambda(x)$.

PART III

9) Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $1 \leq p < r < \infty$. Show that $\|f\| := \|f\|_p + \|f\|_r$ defines a norm on $L^p(X, \mu) \cap L^r(X, \mu)$ making $L^p(X, \mu) \cap L^r(X, \mu)$ into a Banach space

10) In both (a) and (b) we assume that $1 \leq p \leq q < \infty$.
(a) Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Show that if $f \in L^q(X)$ then $f \in L^p(X)$ and $\|f\|_p \leq \mu(X)^s \|f\|_q$ where $s = \frac{1}{p} - \frac{1}{q} = \frac{q-p}{pq}$. (Hint: Show that $|f|^p \in L^{q/p}$ and then use Hölder on suitable functions.)

(b) Let $X = \mathbb{N}$ and let μ be the counting measure on $\mathbb{N}$. Show that if $(a_n) \in L^p$ then $(a_n) \in L^q$ and if $\|(a_n)\|_p \leq 1$ then $\|(a_n)\|_q \leq \|(a_n)\|_p$ with equality if and only if for some j $a_j = 1$ and $a_k = 0$ for all $k \neq j$.

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