Real Analysis **Comprehensive/Qualifying Exam** **August 2022**

The exam has three parts. You must turn in **two** problems from each of parts I and II and **one** problem from part III. All problems have the same weight of twenty points. Only five problems will be graded and counted towards the final grade. Mark the problems you want to be graded. Make sure to have your name clearly written on the solution sheets.

Notation: $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$ denotes the set of extended real numbers,

$$\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\} \quad \text{and} \quad \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}.$$

$(X, \mathcal{A}, \mu)$ will always stand for a measure space. If nothing else is said, then the σ-algebra on $\overline{\mathbb{R}}$, $\mathbb{R}^d$, or subsets thereof will be the Borel σ-algebra and the measure is the Lebesgue measure λ (in case $d = 1$) respectively λ^d (in case $d > 1$). If X is a set and $A \subseteq X$ then χ_A denotes the indicator functions of the set A.

You can use "well known theorem" from the lecture notes or any standard book on measure and integration, **but make sure you state what theorem you are using and make sure you clearly argue that the conditions in the theorem are satisfied**, otherwise you **will not get full credit**.

PART I

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1) Let $Q = \{x \in \mathbb{R}^d \mid (\forall j \in \{1, \ldots, d\})\, 0 \leq x_j \leq 1\}$ and let $f : Q \to \mathbb{R}$ be continuous. Show that

$$G(f) = \{(x, f(x)) \in \mathbb{R}^{d+1} \mid x \in Q\}$$

has Lebesgue measure zero, $\lambda^{d+1}(G(f)) = 0$.

2) Let $E \subset [0, 1]$ be Borel measurable. Show that there exists a Borel measurable subset $A \subset [0, 1]$ such that $\lambda(A) = \frac{1}{2}\lambda(E)$.

3) Let f be defined on a complete measure space $(X, \mathcal{A}, \mu)$, and suppose that for each $\epsilon > 0$ there exists $W \in \mathcal{A}$ such that $\mu(W) < \epsilon$ and $f\chi_{W^c}$ is measurable. Show that f is measurable.

4) a [12 points]) Suppose that $(X, \mathcal{A}, \mu)$ is a finite measure space. Let $f : X \to \overline{\mathbb{R}}$ be a measurable function such that f is finite almost everywhere. Show that for every $\epsilon > 0$ there exists a set $E \in \mathcal{A}$ such that $\mu(E^c) < \epsilon$ and f is bounded on E.

b [4 points]) Give an example that shows that one cannot always find a set $E \in \mathcal{A}$ such that E^c has measure zero and f is bounded on E.

c) [4 points]) Give an example showing that the statement is in general not true if $\mu(X) = \infty$.

PART II

5) Let $f_n : \mathbb{R}_+ \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{n} e^{-x/n}$. Let $0 < a < \infty$.

5-a) Show that there exists a measurable function $f : \mathbb{R}_+ \to \mathbb{R}$ such the sequence $\{f_n\}$ converges to f uniformly on $[0, a]$.

5-b) Does $f_n \to f$ uniformly on $\mathbb{R}_+$?

5-c) Show that

$$\lim_{n\to\infty} \int_{[0,a]} f_n d\lambda \to \int_{[0,a]} f d\lambda \qquad \text{but} \qquad \lim_{n\to\infty} \int_{\mathbb{R}_+} f_n d\lambda \neq \int_{\mathbb{R}_+} f d\lambda.$$

6) Suppose $g : \mathbb{R} \to \mathbb{R}_+$ is continuous, $g(x) = 0$ if $|x| \geq 1$ and $\int g(x) d\lambda(x) = 1$. Show that if $f : \mathbb{R} \to \mathbb{C}$ is continuous then

$$\lim_{n\to\infty} n \int_{\mathbb{R}} g(nx) f(x) d\lambda(x) = f(0).$$

7) 7-a [12 points]) For $f, g \in L^1(\mathbb{R})$ define the convolution of f and g by

$$f * g(x) = \int_{\mathbb{R}} f(y) g(x-y) d\lambda(y)$$

whenever the integral exists. Show that $f * g(x)$ is well defined (and finite) for almost all $x \in \mathbb{R}$ and that $f * g \in L^1(\mathbb{R})$ with $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

7-b [8 points]) Show that if furthermore g is continuous with compact support (that is, there exists $R > 0$ such that $g(x) = 0$ for $|x| > R$) then $f * g$ is finite everywhere and $f * g$ is continuous.

8) Decide if the following limits exists. In case the limit exists find its value.

8-a) $\displaystyle \lim_{n\to\infty} \int_{[1,\infty)} \frac{n}{1 + nx^2} d\lambda(x).$

8-b) $\displaystyle \lim_{n\to\infty} \int_{[0,\infty)} \frac{n}{1 + nx^2} d\lambda(x).$

PART III

9) Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Let $1 \leq p \leq q \leq \infty$. Show that $L^q(X, \mu) \subset L^p(X)$ and that the inclusion map $L^q(X, \mu) \to L^p(X)$, $f \mapsto f$, is bounded. (Hint: Show that $|f|^p \in L^{q/p}(X, \mu)$ and then use Hölder inequality on a suitable function.)

10) Let $I = [0, 1]$. Prove that if $f_n \in L^1(I, \lambda)$ is a such that

$$\sum_{n=1}^{\infty} \int_I |f_n(x)| d\lambda(x) < \infty$$

then $\sum_{n=1}^{\infty} f_n(x)$ converges almost everywhere and

$$\int_I \sum_{n=1}^{\infty} f_n(x) d\lambda(x) = \sum_{n=1}^{\infty} \int_I f_n(x) d\lambda(x).$$

State clearly what theorems you use.