Real Analysis Comprehensive/Qualifying Exam - August 2023

The exam has three parts. You must turn in two problems from each of **Part I** and **Part II** and one problem from **Part III**. All problems have the same weight of twenty points. Only five problems will be graded and counted towards the final grade. Please mark the problems you want to be graded, and make sure to have your name clearly written on the solution sheets.

Notation: $(X, \mathcal{A}, \mu)$ will always stand for a measure space. If nothing else is said, then the σ-algebra on $\mathbb{R}$, $\mathbb{R}^d$, $d \geq 2$, or subsets of these sets will be the Borel σ-algebra, and the measure is the Lebesgue measure λ (in case $d = 1$) (and, respectively, λ^d, $d > 1$).

You can use "well known theorems" from the lecture notes or any standard book on measure and integration, but make sure you state what theorem you are using and make sure you clearly argue that the conditions in the theorem are satisfied, otherwise you will not get full credit.

PART I

(1) For $g : \mathbb{R} \to \mathbb{R}$, let $\|g\|_{L^\infty}$ denote $\sup_{x\in\mathbb{R}} |g(x)|$. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with bounded first and second derivatives.

(a) By considering the Taylor expansion of f, show that

$$\|f'\|_{L^\infty} \leq \frac{1}{h}\|f\|_{L^\infty} + h\|f''\|_{L^\infty}$$

for every $h > 0$.

(b) By minimizing over h in the estimate you showed in (a), show that

$$\|f'\|_{L^\infty} \leq 2\sqrt{\|f\|_{L^\infty}\|f''\|_{L^\infty}}.$$

(2) If E is a measurable subset of $[0, 1]$, prove that there is a measurable subset $A \subset E$ such that $\lambda(A) = \frac{1}{2}\lambda(E)$.

(3) Let f be a nonnegative integrable function on $[0, 1]$. Suppose that for every $n \in \mathbb{N}$,

$$\int_0^1 (f(x))^n \, dx = \int_0^1 f(x) \, dx.$$

Show that f is almost everywhere the characteristic function for some measurable set.

(4) Let f be a nonnegative integrable function on $[0, 1]$ and let $I = \int_0^1 f(x) \, dx$. Show that

$$\sqrt{1 + I^2} \leq \int_0^1 \sqrt{1 + f^2(x)} \, dx \leq 1 + I.$$

(see opposite side for Parts II and III)

PART II

(5) For each $n \geq 3$, let

$$f_n(x) = \begin{cases} n^2 x, & 0 \leq x < \frac{1}{n}, \\ 2n - n^2 x, & \frac{1}{n} \leq x \leq \frac{2}{n}, \\ 0, & \frac{2}{n} < x \leq 1. \end{cases}$$

(a) Sketch the graphs of f_3 and f_4.

(b) Prove that if g is a continuous real-valued function on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 f_n(x) g(x)\, dx = g(0).$$

(Hint: First show that $\int_0^1 f_n(x)\, dx = 1$.)

(6) Show that there is a sequence of Lebesgue measurable functions $f_n : [0,1] \to \mathbb{R}$ that converges pointwise everywhere to a function f having the property that there is not a subset A of $[0,1]$ of Lebesgue measure 0 such that f_n converges uniformly to f on the complementary set $[0,1] \setminus A$. Write a short paragraph commenting on the comparison of the result of this exercise with the statement of Egoroff's theorem.

(7) Assume that the real-valued measurable function $f : (t,x) \mapsto f(t,x)$ and its partial derivative $\frac{\partial f}{\partial t}$ are bounded on $[0,1]^2$. Show that for $t \in (0,1)$,

$$\frac{d}{dt}\left[\int_0^1 f(t,x)\, dx\right] = \int_0^1 \frac{\partial f}{\partial t}(t,x)\, dx.$$

(Hint: Consider the difference quotient associated to the derivative on the left.)

(8) Let λ, μ, ν be nonnegative measures on a σ-algebra $\mathcal{A} \subset \mathscr{P}(X)$ which have the relationship $\lambda \ll \mu \ll \nu$. Suppose that λ and μ are finite measures and that ν is a σ-finite measure.

(a) Show that $\lambda \ll \nu$.

(b) Show that

$$\frac{d\lambda}{d\nu} = \frac{d\lambda}{d\mu}\frac{d\mu}{d\nu}.$$

(Hints: Let $f = \frac{d\lambda}{d\mu}$, $g = \frac{d\mu}{d\nu}$, and $h = \frac{d\lambda}{d\nu}$, and show that there is a sequence f_n of special simple functions such that $f_n \nearrow f$ pointwise almost everywhere. Show that $|\lambda(E) - \int_E f_n\, d\mu| \to 0$ for all $E \in \mathcal{A}$ as $n \to \infty$. Show that $\int_E f_n\, d\mu = \int_E f_n g\, d\nu$ for all $E \in \mathcal{A}$. Now, use the essential uniqueness of the Radon-Nikodym derivative to complete the proof that $f_n g \to h$.)

PART III

(9) Let f_n be a sequence of continuous functions on $[0,1]$ that converges pointwise for all $x \in [0,1]$. Given $\epsilon > 0$, show that there is a subset $A \subset [0,1]$ with $\lambda(A) < \epsilon$ and a positive real number $M > 0$ such that

$$|f_n(x)| \leq M$$

for all $x \in [0,1] \setminus A$ and all $n \geq 1$.

(10) Let $(X, \mathcal{A}, \mu)$ be a measure space. Recall that $L^1(X) = L^1(X; d\mu)$ is the set of integrable functions on X.

(a) Does $f \in L^1(X)$ imply $\displaystyle\lim_{\lambda\to+\infty} \int_{\{x:|f(x)|>\lambda\}} |f| d\mu = 0$? If so, give a proof; if not, give a counterexample showing that the claim can fail.

(b) Does $\displaystyle\lim_{\lambda\to+\infty} \int_{\{x:|f(x)|>\lambda\}} |f| d\mu = 0$ imply $f \in L^1(X)$? If so, give a proof; if not, give a counterexample showing that the claim can fail.