Mathematics Comprehensive Examination
Core 1: Analysis
January 2004

Instructions: This exam is divided into three parts: Part A, Part B, and Part C. Follow the instructions for each part. Each problem submitted for evaluation must be written on a separate sheet of paper with your name at the top. Each problem will be scored out of 10 points. Your answers must be complete, concise, and clear. You should quote theorems and other well known results that you use. Make sure they are properly used. *All references to measure and integration refer to Lebesgue measure and integration on the real line.* You have $2\frac{1}{2}$ hours to complete this exam. We wish you well.

Part A: Answer A.1 and A.2

(A.1) Select two of the following three well known results and precisely state its hypotheses and conclusions.

a. Egoroff's Theorem

b. Lebesgue's Dominated Convergence Theorem

c. Jensen's Inequality

(A.2) Answer each statement below true or false. Do not justify.

a. If f is a measurable function on $\mathbb{R}$ then the inverse image of a measurable set is measurable.

b. If $1 < p < q < \infty$ then $L^q[0,1] \subset L^p[0,1]$.

c. If $f : [0,1] \to \mathbb{R}$ is of bounded variation then f is differentiable a.e. and for each $x \in [0,1]$, $f(x) - f(0) = \int_0^x f'(t)\,dt$.

d. If $f \in L^2[0,1]$ and $\epsilon > 0$ there is a step function g such that $\|f - g\|_2 < \epsilon$.

e. If f_n is a sequence of measurable functions on $[a,b]$ and $f_n \to f$ a.e. then $f_n \to f$ in measure.

Part B: Prove two of the following three statements.

(B.1) If E is a measurable subset of $[0,1]$ then there is a measurable subset $A \subset E$ such that $m(A) = \frac{1}{2}m(E)$. (Hint: Consider the function $f(x) = m(E \cap [0,x])$).

(B.2) Prove the following:

$$\lim_{n\to\infty} \int_0^\infty \frac{\cos(nx)}{\sqrt{x}\sqrt{1+n^2x^2}}\,dx = 0.$$

(B.3) Let f be a nonnegative real function on $[0,1]$ and let $I = \int_0^1 f(x)\,dx$. Show that

$$\sqrt{1+I^2} \le \int_0^1 \sqrt{1+f^2(x)}\,dx \le 1 + I.$$

(Hint: For the first inequality consider using Jensen's inequality)

Part C: Prove two of the following three statements.

(C.1) Let f_n be a sequence of continuous functions on $[0,1]$ that converges point-wise for all $x \in [0,1]$. Given $\epsilon > 0$ show there is a subset $A \subset [0,1]$, $m(A) < \epsilon$, and a positive number M such that

$$|f_n(x)| \leq M,$$

for all $x \in [0,1] \setminus A$.

(C.2) Let f be a continuous function on $[0,1]$. Find the following limits:

a. $\lim_{n\to\infty} \int_0^1 x^n f(x)\,dx$.

b. $\lim_{n\to\infty} n \int_0^1 x^n f(x)\,dx$.

(Hint: The Weierstrass approximation theorem may be helpful for part b).

(C.3) Let $\sim$ be the equivalence relation on $\mathcal{M}$, the set of measurable subsets of $\mathbb{R}$, defined by $A \sim B$ if $m(A \triangle B) = 0$, where $A \triangle B$ is the symmetric difference of A and B. Let $X = \mathcal{M}/\sim$ denote the set of equivalence classes. On X we define a metric d by

$$d(A,B) = m(A \triangle B).$$

Suppose A_n is a sequence in X. Show that $\lim_{n\to\infty} d(A_n, A) = 0$ if and only if χ_{A_n} converges to χ_A in measure.