Real Analysis   Comprehensive/Qualifying Exam   January 2022

The exam has three parts. You must turn in **two** problems from each of parts I and II and **one** problem from part III. All problems have the same weight. Only five problems will be graded and counted towards the final grade. Mark the problems you want to be graded. Make sure to have your name clearly written on the solution sheets.

Notation: $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$ denotes the set of extended real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$. $(X, \mathcal{A}, \mu)$ will always stand for a measure space. The space of measurable functions is denoted by $\mathcal{M}(X)$. If nothing else is said, then the σ-algebra on $\overline{\mathbb{R}}, \mathbb{R}^d$, or subsets thereof will be the Borel σ-algebra and the measure is the Lebesgue measure λ (in case $d = 1$) respectively λ^d. If X is a set and $A \subseteq X$ then χ_A denotes the indicator functions of the set A.

You can use "well known theorem" from the lecture notes or any standard book on measure and integration, **but make sure you state what theorem you are using and make sure you clearly argue that the conditions in the theorem are satisfied**, otherwise you **will not get a full credit**. The exception if you are asked to prove one of those statements.

PART I

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1) Let $E \subset \mathbb{R}$ be measurable set of finite measure with the property that

$$\lambda(E \cap I) \leq \frac{1}{2}\,\lambda(I)$$

for every open interval $I \subset \mathbb{R}$. Show that $\lambda(E) = 0$.

2) Let $p(x, y)$ be a polynomial in two variables, but not the zero polynomial. Prove that the set of points $(x, y) \in \mathbb{R}^2$ with $p(x, y) = 0$ has a 2-dimensional Lebesgue measure zero, $\lambda^2(p^{-1}(0)) = 0$.

3) Let $f \in L^1(X, \mu)$, $f \geq 0$. Define the measure $\mu_f : \mathcal{A} \to [0, \infty)$ by

$$\mu_f(A) = \int_A f d\mu \qquad A \in \mathcal{A}.$$

Show that if $g \in \mathcal{M}(X)$ then $g \in L^1(X, \mu_f)$ if and only if $fg \in L^1(X, \mu)$ and in that case

$$\int_X g d\mu_f = \int_X f g d\mu.$$

4) Let $f : X \to \overline{\mathbb{R}}$ be a measurable function on a finite measure space $(X, \mathcal{A}, \mu)$. Suppose that f is finite for almost all x. Prove that for every $\epsilon > 0$ there exists a set $A \in \mathcal{A}$ with $\mu(X \setminus A) < \epsilon$, such that f is bounded on A. Give an example for which it is impossible to require that A^c is a null set.

PART II

5) Assume that X is a countable set and that μ is the counting measure. Let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions and let $f : X \to \mathbb{R}$. Show that $f_n \to f$ in measure if and only if $f_n \to f$ uniformly.

6) Compute the following limits:

a) $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}_+} \frac{\sin x}{1+nx^2}\, d\lambda(x).$

b) $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}_+} \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)\, d\lambda(x).$

7) For each of the following two integrals check whether the limit exists. If it exists, find its value.

a) $\displaystyle\lim_{n\to\infty} \int_1^n \left(1-\frac{x}{n}\right)^n d\lambda(x).$

b) $\displaystyle\lim_{n\to\infty} \int_1^{2n} \left(1-\frac{x}{n}\right)^n d\lambda(x).$

8) Let $\{f_n\}$ be a sequence in $L^1(X,\mu)$ and assume that there exists a measurable function f such that $f_n \to f$ uniformly. Show the following:

a) If $\mu(X) < \infty$ then $f \in L^1(X,\mu)$ and $f_n \to f$ in L^1.

b) Give an example of a sequence $\{f_n\}$ in $L^1(\mathbb{R})$ such that $f_n \to f$ uniformly but the sequence $\{f_n\}$ does not converge to f in L^1.

PART III

9) Let $\mathbf{X}$ be normed vector space and let $\mathbf{Y}$ be a Banach space. Denote by $\mathrm{B}(\mathbf{X},\mathbf{Y})$ the vector space of bounded linear maps $T : \mathbf{X} \to \mathbf{Y}$. Show that $\mathrm{B}(\mathbf{X},\mathbf{Y})$ with the norm

$$\|T\| = \sup\{\|Tv\| \mid v \in \mathbf{X},\ \|v\| = 1\}$$

is a Banach space.

10) Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Let $f : X \to \mathbb{R}$ be measurable. Show that

$$\lim_{n\to\infty} \int_X \cos^{2n}(f(x))d\mu(x) = \mu(\{x \in X \mid f(x) \in \mathbb{Z}\})$$

where $\mathbb{Z}$ is the set of integers.

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