Directions: There are five parts: Parts A-E. Do one problem from each part. Thus exactly five problems are to be handed in. All problems have equal weight. Each problem submitted must be written on a separate sheet of paper with your name and problem number at the top. Your solutions must be complete and clear. You must show that the hypotheses of well known results that you use are satisfied. Make sure they are properly used and quoted. All references to measure and integration are in the sense of Lebesgue. You have $2\frac{1}{2}$ hours. We wish you well.

Part A

A1 Recall that a step function is a finite linear combination of characteristic functions of intervals.

i. Show that every continuous function on $[0,1]$ is a uniform limit of step functions.

ii. Is the converse true? (Justify your answer)

A2 Prove: If $f \in C[0,1]$ and $\int_0^1 f(x)e^{-nx}\,dx = 0$ for all $n = 1, 2, 3, \ldots$, then $f = 0$.

Part B

B1 Prove that, if there exists a G in G_δ with $E \subset G$ and $m^*(G \setminus E) = 0$, then E is measurable. (Here m^* is outer measure)

B2 Let E be a measurable set in $[0,1]$ and let $c > 0$. If $m(E \cap I) \geq cm(I)$, for all open intervals $I \subset [0,1]$, show that $m(E) = 1$.

Part C

C1 Show there are no bounded sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ for which

$$f_n(x) = a_n \sin(2\pi nx) + b_n \cos(2\pi nx)$$

converges to 1 almost everywhere on $[0,1]$. Here $\mathbb{N}$ is the set of natural numbers.

C2 Let $f \in L^1(0,\infty)$ and suppose $\int_0^\infty x|f(x)|\,dx < \infty$. Prove that the function

$$g(y) = \int_0^\infty e^{-xy} f(x)\,dx$$

is differentiable at every $y \in (0,\infty)$.

Part D

D1 Prove that, if f is differentiable a.e. on $[0,1]$ and f' is not in $L^1([0,1])$, then f is not of bounded variation on $[0,1]$.

D2 Show that the product of two absolutely continuous functions on a closed finite interval $[a,b]$ is absolutely continuous.

Part E

E1 Let X be the normed linear space obtained by putting the norm $\|f\|_1 = \int_0^1 |f(x)|\,dx$ on the set of real continuous functions.

i. Show that X is not a Banach space.

ii. Show that the linear functional $\Lambda f = f(1/2)$ is not bounded.

E2 Suppose $1 < p < q < r < \infty$. Show that $L^p(\mathbb{R}) \cap L^r(\mathbb{R}) \subset L^q(\mathbb{R})$.