MATHEMATICS COMPREHENSIVE EXAMINATION
CORE 1-ANALYSIS
SPRING-2007

DIRECTIONS: This test consists of two parts (A) and (B). Choose *one problem from Part* (A) and *four problems from Part* (B). Please answer the problems in the order in which they appear and turn in only those that you wish to have graded. Be sure to refer to all theorems you apply and check hypotheses are satisfied. You have two and one half hours for this test. We wish you well.

TERMINOLOGY: The integrals that appear in this examination are to be understood as Lebesgue integrals. Lebesgue measure on $\mathbb{R}$ or $\mathbb{R}^d$ is denoted by m.

PART A

A.1 Let $f : \mathbb{R}^d \to [0, \infty]$ be Lebesgue measurable. Show μ defined by $\mu(E) = \int_E f \, dm$ is a measure on the Lebesgue measurable sets and μ is σ-finite if and only if $f(x) < \infty$ for a.e. x. Recall μ is σ-finite if there is a sequence E_n, $n = 1, 2, \ldots$ of sets of μ finite measure whose union is $\mathbb{R}^d$.

A.2 Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable and let $h : \mathbb{R}^d \to \mathbb{R}$ be Lebesgue measurable. In general, it is known that $f \circ h$ need not be Lebesgue measurable. However, show if the preimage under h of any Borel subset E of Lebesgue measure 0 has Lebesgue measure 0 in $\mathbb{R}^d$, then $f \circ h$ is Lebesgue measurable.

PART B

B.1 Let $f : \mathbb{R} \to [-\infty, \infty]$ be a measurable function. Show if $t_0 \in \mathbb{R}$, then the function F defined by $F(t) = f(t + t_0)$ is measurable.

B.2 Determine

$$\lim_{n\to\infty} \int_1^{\infty} \sin\left(\frac{x}{n}\right) \frac{n^3}{1 + n^2 x^3} \, dx.$$

B.3 Let $f : \mathbb{R}^d \to \mathbb{R}$ be Lebesgue integrable. Suppose $x_n \in \mathbb{R}^d$ for $n = 1, 2, 3, \ldots$ satisfies $|x_n| \to \infty$ as $n \to \infty$. Show

$$\int_{[0,1]^d} f(x + x_n) dm(x) \to 0$$

as $n \to \infty$.

B.4 Let $f \in L^1(\mathbb{R}^d)$ and suppose x_n, $n = 1, 2, 3, \ldots$ is a sequence in $\mathbb{R}^d$. Show

$$H(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x + x_n)$$

converges for a.e. x and is an integrable function.

B.5 Suppose $H(x, y) = f(x) + f(y)$ is integrable on $[0, 1] \times [0, 1]$ and

$$\int_{[0,1]^2} H(x, y)\, dm(x, y) = 12.$$

Show f is integrable on $[0, 1]$ and find $\int_0^1 f(x)\, dx$.

B.6 Assume $f : [a, b] \to \mathbb{R}$ is absolutely continuous. Show $|f|$ is absolutely continuous.