Instructions: Do ***five (5)*** *of the 8 problems, including* ***at least two (2) from Part A*** and ***at least two (2)*** *from* ***Part B****. Start each chosen problem on a fresh sheet of paper.* **Write your name at the top of each sheet**. **Please turn in 5 problems, even if the solutions are imperfect, for partial credit.** Logical rigor is important. Describe which theorems you apply, checking that all hypotheses are satisfied. If you see a gap in your proof but not how to fix it, state this because it is better that you recognized it! *Clip your papers together in numerical order of the problems chosen when finished.* You have 3 hours. **Good luck!**

Symbols: Integration is in the sense of Measure and Integration theory. Lebesgue measure on $\mathbb{R}$ or $\mathbb{R}^n$ is l, and $(X, \mathfrak{A}, \mu)$ is an abstract measure space. Also, 1_S is the *characteristic*, or *indicator*, function of a set S.

Part A: Measures

1. If E is a Lebesgue measurable subset of $[0,1]$ prove carefully that there is a measurable subset $A \subset E$ such that $l(A) = \frac{1}{2}l(E)$.

2. Let $f : X \to \mathbb{R}^*$ be an extended real-valued measurable function on a finite measure space $(X, \mathfrak{A}, \mu)$. Suppose that $f(x)$ is finite for almost all x.

   (a) Prove that for each $\epsilon > 0$ there exists a set $A \in \mathfrak{A}$, with $\mu(X \setminus A) < \epsilon$, such that f is bounded on A.

   (b) Give an example for which it is impossible to require that $X \setminus A$ be a μ-null set.

3. Suppose $f_n : X \to \mathbb{R}$ is a measurable function for each $n \in \mathbb{N}$, where $(X, \mathfrak{A}, \mu)$ is a measure space. Prove that the set $S = \left\{ x \,\middle|\, \lim_{n\to\infty} f_n(x) \text{ exists} \right\}$ is a measurable set.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be *Lebesgue* measurable. Prove that f is equal almost everywhere to a *Borel* measurable function h. Hint: The function f is the pointwise limit of a sequence $\phi_n \in \mathfrak{S}$, the set of simple functions. Modify the functions ϕ_n suitably.

Part B: Integrals

5. Consider the sequence of functions $f_n(x) := 1_{[-n,n]}(x)\sin\left(\frac{\pi x}{n}\right)$ for all $x \in \mathbb{R}$.

   (a) Determine $f(x) = \lim_{n\to\infty} f_n(x)$ and show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{R}$. Does the sequence converge uniformly on $\mathbb{R}$?

   (b) Show that $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\,dx$. Are the assumptions of Lebesgue's dominated convergence theorem satisfied?

6. Let $f(x)$ be a real-valued measurable function on a finite measure space $(X, \mathfrak{A}, \mu)$. Show that $\displaystyle\lim_{n\to\infty}\int_X \cos^{2n}(\pi f(x))\,d\mu = \mu\{x \mid f(x) \in \mathbb{Z}\}$ where $\mathbb{Z}$ is the set of integers.

7. Suppose f and h are in $L^1(\mathbb{R})$, and let $H(x,y) = f(x-y)h(y)$. Show that $H \in L^1\left(\mathbb{R}^2\right)$. Use this to show the function $f * h(x) = \displaystyle\int_{\mathbb{R}} f(x-y)h(y)\,dy$ is defined almost everywhere and is an integrable function on $\mathbb{R}$. Then show that $\|f*h\|_1 \le \|f\|_1\|h\|_1$. (Hint: Use Fubini's Theorem. You may *assume* the measurability of H over the plane.)

8. Do *either* part (a) *or* part (b) but *not* both: *Credit given for* ***one*** *part only!*

   (a) Suppose that both f and $\frac{\partial f}{\partial y}$ lie in $L^1\big([a,b]\times[c,d]\big)$ and that $f(x,y)$ is absolutely continuous as a function of y for almost all fixed values of x. Prove $\displaystyle\frac{d}{dy}\int_a^b f(x,y)\,dl(x)$ exists and equals $\displaystyle\int_a^b \frac{\partial f}{\partial y}(x,y)\,dl(x)$ for almost all y. *Hint:* Prove that g is constant where $g(y) = \displaystyle\int_a^b f(x,y)\,dl(x) - \int_c^y \left(\int_a^b \frac{\partial f}{\partial t}(x,t)\,dl(x)\right)dl(t)$.

   (b) Let $\phi : [0,1] \to [0,1]$ be the Cantor function. Let $f(x,y) = \phi(x+y)1_D(x,y)$ for all $(x,y) \in D = \{(x,y) \in \mathbb{R}^2 \mid x \ge 0,\ y \ge 0,\ x+y \le 1\}$. Show that $\displaystyle\frac{d}{dy}\int_{\mathbb{R}} f(x,y)\,dl(x) \ne \int_{\mathbb{R}} \frac{\partial}{\partial y} f(x,y)\,dl(x)$. Reconcile this with the claim in part (a).