Introduction to Applied Mathematics
Qualifying Exam, January 2024

The exam is 3 hours. Each problem is worth 20 points, and you can do a maximum of five problems for a total potential score of 100 points. Pick at least one from each category.

Category I: Continuum Mechanics

1. Suppose we had a system with a fixed uniform rate of mass decay described as follows. Let $x = \varphi(X, t) = \varphi_t(X)$ be the map from material coordinates to spatial coordinates, and let $\rho(x, t)$ be the spatial mass density field and let $v(x, t) = \frac{\partial}{\partial t}\varphi(X, t)$. Assume all fields are smooth. Suppose for $\Omega \subset B$ open where B is the body and $\Omega_t = \varphi_t(\Omega)$ that we have the mass function

$$\text{mass}[\Omega_t] = \int_{\Omega_t} \rho(x, t) dV_x.$$

Suppose

$$\frac{d}{dt}\text{mass}[\Omega_t] = -\int_{\Omega_t} \gamma(x, t)\rho(x, t) dV_x \tag{1}$$

for some smooth function $\gamma(x, t) > 0$. From the mass relation in (1), derive the PDE in spatial coordinates

$$\frac{\partial}{\partial t}\rho(x, t) + \nabla^x \cdot (\rho(x, t)v(x, t)) = -\gamma(x, t)\rho(x, t), \qquad x \in \varphi_t(B),\ t \geq 0. \tag{2}$$

2. Consider the linear balance law in spatial coordinates for constant density

$$\rho_0 \frac{d}{dt} v = \nabla \cdot S + \rho_0 b$$

for body force field b, Cauchy stress tensor S, density $\rho_0 \neq 0$, and velocity v. $\frac{d}{dt}$ is defined as the partial derivative in time for fields in material coordinates. Assume all fields are smooth. Assume there is a smooth scalar pressure field $p(x, t)$ and I is the identity matrix such that

$$S = -pI + \mu(\nabla^x v + \nabla^x v^T),$$

and assume $\nabla^x \cdot v = 0$, the incompressibility condition.

(a) **(10 points)** Derive the equation:

$$\rho_0[\partial_t v + (\nabla^x v)v] = \mu \triangle^x v - \nabla^x p + \rho_0 b.$$

(b) **(10 points)** Suppose $v(x,t) = (0,0,v_3(x_1,x_2,t))^T$, $b = 0$, and $p(x) = p_0 > 0$ for $x = (x_1,x_2,x_3)$. For fixed x_3, derive a scalar-valued PDE in x_1, x_2 describing the dynamics of the fields v and p.

3. Consider the electric D field arising from a single charge living at $y \in \mathbb{R}^3$ defined by

$$D_y(x) = k\frac{(x-y)}{|x-y|^3}.$$

Consider for $y \in \mathbb{R}^3$ the delta distribution $\delta_y \in \mathcal{D}'(\mathbb{R}^3)$ defined by

$$\langle \delta_y, \phi \rangle = \phi(y), \qquad \phi \in C_c^\infty(\mathbb{R}^3).$$

(a) Derive in the distributional sense $\nabla \cdot D_y = k\delta_y$, and find k.

(b) If $D_N = \frac{1}{N}\sum_{i=1}^N D_{(i/N,0,0)}$ and $\rho_N = \frac{k}{N}\sum_{i=1}^N \delta_{(i/N,0,0)}$, derive distributions D and ρ such that $D_N \to D$ and $\rho_N \to \rho$. Prove $\nabla \cdot D = \rho$ in the distributional sense.

Category II: Fourier Analysis

4. Consider the heat equation $\frac{\partial}{\partial t}u(x,t) = \triangle u(x,t)$, and suppose $u(x,0) = u_0(x)$ where $u_0 \in \mathcal{S}(\mathbb{R}^d)$. Here $u(x,t)$ is a scalar field.

a. (10 points) Find the classical solution $u(x,t)$ and verify that it is a classical solution, i.e. that u is twice continuously differentiable in x and continuously differentiable in t, and satisfies the PDE and initial data.

b. (10 points) Now suppose $u_0 \in L^2(\mathbb{R}^d)$. Use the same formula for your solution as in part (a) and show that $u(x,t)$ satisfies classically the PDE $\frac{\partial}{\partial t}u(x,t) = \triangle u(x,t)$ for $t > 0$.

5. Consider $\sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ smooth with sufficiently tempered growth. Suppose we define an operator Q acting on $\mathcal{S}(\mathbb{R}^d)$ by

$$Q\psi(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \sigma(x,\xi)\psi(y)e^{2\pi i(x-y)\cdot\xi}dyd\xi. \tag{3}$$

a. (10 points) Show that if $\sigma(x,\xi) = q(x) + \xi^T A\xi$ for A a $d \times d$ positive definite matrix, then Q can be written as a linear differential operator. Find that linear differential operator.

b. (10 points) Suppose $Q\psi(x) = g(x)\cdot\nabla\psi(x)$ for some vector-valued continuous and bounded function $g(x)$. Find $\sigma(x,\xi)$ that satisfies Equation (3).

6. Let f be smooth and periodic, and consider the ODE

$$-\frac{d^2}{dx^2}u(x) + \frac{2}{i}\frac{d}{dx}u(x) + u(x) = f(x).$$

Find the solution $u(x)$ in terms of the Fourier modes $\hat{f}(n) = \int_0^1 e^{2\pi i n x} f(x)dx$, and prove u is smooth and periodic.

Category III: Weak-form PDEs & Distribution Theory

7. Derive the distributional derivative of $\ln|x|$ in $\mathcal{D}'(\mathbb{R})$.

8. Let $\Omega = \{x \in \mathbb{R}^3 : |x| \in (1,2)\}$ and let $H_0^1(\Omega)$ be defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\phi\|_1^2 = \int |\nabla\phi|^2 dx$. Consider the following PDE for $f \in L^2(\Omega)$:

$$\begin{aligned} -\triangle u(x) + |x|^2 u(x) &= f(x), && x \in \Omega,\\ u(x) &= 0, && x \in \partial\Omega. \end{aligned}$$

Formulate the PDE in the weak sense, and prove the existence of a solution $u \in H_0^1(\Omega)$.

9. **a. (10 points)** Prove the identity for $\Phi, \Psi \in C^1(\mathbb{R}^3;\mathbb{R}^3)$:

$$\nabla\cdot(\Psi\times\Phi) = \Phi\cdot(\nabla\times\Psi) - (\nabla\times\Phi)\cdot\Psi.$$

b. (10 points) Consider simply connected domain $\overline{\Omega} = \overline{\Omega_1\cup\Omega_2}$, where Ω_1 and Ω_2 are disjoint simply connected open sets. Let Γ be the boundary between Ω_1 and Ω_2. Consider the curl operator in the sense of distributions denoted $\overline{\nabla}\times$ on $C_c^\infty(\Omega;\mathbb{R}^3)$. Let F be defined such that $F(x) = F_j(x)$ for $x \in \Omega_j$ where $F_j \in C^2(\overline{\Omega_2};\mathbb{R}^3)$. Let $[F](x) := F_2(x) - F_1(x)$ for $x \in \Gamma$. Let $N(x)$ be the normal vector for $x \in \Gamma$ aiming into Ω_2. Prove

$$\overline{\nabla}\times F = \nabla\times F_1\Big|_{\Omega_1} + \nabla\times F_2\Big|_{\Omega_2} + N\times[F]\delta_\Gamma.$$

For a function $G(x)$, the distribution $G\delta_\Gamma$ is defined such that $\langle G\delta_\Gamma, \Phi\rangle = \int_\Gamma G\cdot\Phi$.