Instructions: [*You must work two problems from each section for a total of four problems. Only your first four solutions will be graded.*]. *Be sure to write the number for each problem with your work and write your name clearly at the top of each page you turn in for grading. You have three hours.* Good luck!

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1 Point Set Topology

1. (a) Prove that for $A \subset X$, a topological space, $\overline{A} \setminus \operatorname{Int} A = \overline{A} \cap \overline{X \setminus A}$.
   (b) Prove or disprove: If A is a connected subset of X and if $A \subseteq D \subseteq \overline{A}$, then D is connected.
   (c) Prove or disprove: If A is connected, then its interior A° is connected.
2. Let $p\colon X \to Y$ be a quotient map. Show that if Y is connected and $p^{-1}(y)$ is connected for each $y \in Y$, then X is connected.
3. Let $f : X \to Y$ by a continuous map.
   - If A is compact, prove that $f(A)$ is compact.
   - If A is connected, prove that $f(A)$ is connected.
4. Let $\mathcal{H}([0,1])$ be the set of all non-empty closed subsets of the interval $[0,1]$.
   - Show that $d(A,B) = \max_{a\in A}(\inf_{b\in B}|a-b|)$ is a metric on $\mathcal{H}([0,1])$.
   - Given the property that this metric is complete, prove that is a closed subset such that $T(C) = C$, where $T(C) = T_1(C) \cup T_2(C)$ where $T_1(x) = x/3, T_2(x) = x/3 + 2/3$. (Hint: Show that the function $T : \mathcal{H}([0,1]) \to \mathcal{H}([0,1])$ has a fixed point.)

2 Homotopy

5. Show that the following three conditions (on a topological space X) are equivalent:
   (a) Every map $S^1 \to X$ is homotopic to a constant map.
   (b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.
   (c) The fundamental group, $\pi_1(X, x_0)$, is trivial for all $x_0 \in X$.
6. State precisely the Seifert-van Kampen theorem and use it to compute the fundamental group of the connected sum of two projective planes, $\mathbb{R}P^2$ (equivalently, the space obtained by identifying two Möbius bands along the boundary circle). Describe all the regular covering spaces.
7. Let $X = \{(x,y,z) \in \mathbb{R}^3 | x = 0;\ y = 0\} \cup \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 = 1;\ z = 0\}$. Calculate the fundamental group of $\mathbb{R}^3 \setminus X$.

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8. Show every continuous map from the real projective plane to a torus is homotopic to a constant map. Give an example of a continuous map from a torus to the real projective plane which is not homotopic to a constant map. Explain why your example is not homotopic to a constant map.