**Instructions**:    [*You must work two problems from each section and one additional question from either section for a total of **five** problems. Only your **first five** solutions will be graded.*]. *Be sure to write the number for each problem with your work and write your name clearly at the top of each page you turn in for grading. You have three hours.*

# 1   Point Set Topology

1. Compact sets:

   (a) Prove that the continuous image of a compact set is compact.

   (b) Must the continuous image of a closed set be closed? (Justify your answer.)

   (c) Prove that a compact subset of a Hausdorff space is closed.

2. Let $X$ and $Y$ be topological spaces and let $f : X \to Y$ by a functions. Prove that the following are equivalent:

   (a) For all open $U$ in $Y$, $f^{-1}(U)$ is open in $X$.

   (b) For all open $F$ in $Y$, $f^{-1}(F)$ is closed in $X$.

   (c) For all $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$.

3. Let $X$ be compact, $Y$ Hausdorff and connected and let $f : X \to Y$ be a continuous open mapping. Prove that $f$ is surjective.

4. Let $X$ be a topological space and let $f, g : X \to \mathbb{R}$ be continuous. *Use the pasting lemma* to prove that $h(x) = \min\{f(x), g(x)\}$ is continuous.

# 2   Homotopy

5. Let $p : E \to B$ be a covering map.

   (a) Prove that $E$ is Hausdorff if $B$ is Hausdorff

   (b) Prove that $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0 = p(e_0))$ is injective.

6. Let $X$ be a topological space with basepoint, $x_0 \in X$ and let $\{U, V\}$ be an open cover satisfying the conditions of the Seifert-van Kampen theorem. If the inclusion of $U \cap V$ into $U$ induces an isomorphism of fundamental groups, show there is a homomorphism $\Phi : \pi_1(X, x_0) \to \pi_1(V, x_0)$ extending the identity on $\pi_1(V, x_0)$.

7. Let $S^n$ denote the unit $n$-sphere, $\{v = (x_1, x_2, \ldots, x_{n+1}) | x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1\}$ in $\mathbb{R}^{n+1}$. Define an equivalence relation on $S^n$ by $v \sim w \iff v = \pm w$. Verify (for all $n \geq 0$) whether or not the quotient space $S^n/\sim$ is homeomorphic to $S^n$.

8. Consider the subset $E$ of $\mathbb{R}^2$ consisting of the union of the sets $[i - 1/4, i + 1/4] \times \mathbb{R}$ and $\mathbb{R} \times i - 1/4, i + 1/4]$ over all integers $i$. Consider the equivalence relation: $(x, y) \sim (u, v)$ iff $(x - u, y - v) \in \mathbb{Z}^2$. Show that $p : E \to B = E/\sim$ is a covering map. Compute the fundamental group of $B$.