**Instructions**: *Work 2 problems from section A, 2 problems from section B, and 1 problem from section C.* Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours.

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## A Point Set Topology (2 problems)

**A1.** Let $f\colon X \to Y$ be a continuous function between topological spaces $X$ and $Y$, and let $A$ be a subset of $X$.

(a) If $A$ is compact, prove that $f(A)$ is compact.

(b) If $A$ is connected, prove that $f(A)$ is connected.

**A2.** Let $p\colon X \to Y$ be a quotient map, and $f\colon X \to Z$ a continuous function such that $f(x_1) = f(x_2)$ whenever $p(x_1) = p(x_2)$. Show that there is a unique function $g\colon Y \to Z$ such that $g \circ p = f$, and show that $g$ is continuous.

**A3.** A metric $d$ on a set $X$ is called *bounded* if there is a positive real constant $M$ so that $d(x, y) \leq M$ for any pair of points $x, y$ in $X$. Show that given any metric $\delta$ on a set $X$, there is a bounded metric $d$ on $X$ that induces the same topology as $\delta$.

## B Homotopy (2 problems)

**B1.** Suppose that $X$ and $Y$ are topological spaces, and let $x_0 \in X$ and $y_0 \in Y$. Prove that $\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

**B2.** Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n > 2$.

**B3.** Let $p\colon E \to B$ be a covering map, where $E$ and $B$ are path-connected and locally path-connected, and let $p(e_0) = b_0$. Show that $H = p_*(\pi_1(E, e_0))$ is a normal subgroup of $\pi_1(B, b_0)$ if and only if for each pair of points $e_1, e_2$ of $p^{-1}(b_0)$ there is an equivalence of covering spaces $h\colon E \to E$ with $h(e_1) = e_2$.

## C Mixed (1 problem)

**C1.** Let $A \subset X$ be a retract: there is a continuous map $r\colon X \to A$ such that $r(a) = a$ for each $a \in A$.

(a) If $X$ is Hausdorff, show that $A$ is closed.

(b) If $a_0 \in A$, show that the homomorphism $r_*\colon \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective.

**C2.** Let $p\colon E \to B$ be a covering map, where $B$ is connected and path-connected.

(a) If $B$ is compact, and $p^{-1}(b)$ is finite for each $b \in B$, show that $E$ is compact.

(b) If there is a point $b_0 \in B$ so that $p^{-1}(b_0)$ has $k$ elements, show that $p^{-1}(b)$ has $k$ elements for every $b \in B$.