Instructions: *Work two problems from section A, two problems from section B, and one additional problem from either section, for a total of five problems.* Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours. Good luck!

Section A.

A1 Let X and Y be topological spaces, and $f: X \to Y$ a function. Recall that "f is open" means: if U is an open set in X then $f(U)$ is open in Y.

(a) If f is continuous, does it follow that f is open? (Proof or counterexample.)

(b) If f is open, does it follow that f is continuous? (Proof or counterexample.)

(c) Show that if $X = Y \times Z$ (a product of topological spaces, with the product topology) and f is the projection on Y, then f is open.

(d) Under the conditions and notation of (c), if F is closed in $X = Y \times Z$, does it follow that $f(F)$ is closed in Y? (Proof or counterexample.)

A2 Recall that a metric d on a set X is called *bounded* if there is a positive real constant M such that $d(x, y) \leq M$ for any pair of points x, y in X. Show that given any metric δ on a set X, there is a bounded metric d on X that induces the same topology as δ.

A3 Let $Y = \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$ and let Z be the graph of the function $y = \sin(\pi/x)$ for $0 < x \leq 1$. Is the set $X = Y \cup Z$ connected or disconnected in the standard topology on $\mathbb{R}^2$? Prove your answer.

A4 Let $f: S^1 \to \mathbb{R}$ be continuous, where S^1 is the unit circle in $\mathbb{R}^2$.

(a) Show that there is a point $z \in S^1$ such that $f(z) = f(-z)$.

(b) Show that f is not surjective.

Section B.

B1 Show that the following three conditions on a topological space X are equivalent:

(a) Every map $S^1 \to X$ is homotopic to a constant map.

(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

(c) The fundamental group $\pi_1(X, x_0)$ is trivial for all $x_0 \in X$.

B2 Prove the Brouwer fixed point theorem in dimension two: every continuous map $f: D^2 \to D^2$ has a fixed point.

B3 State the Seifert-Van Kampen Theorem. Use this theorem to show that the n-sphere S^n is simply connected for $n \geq 2$.

B4 Let X be path-connected and locally path-connected, and suppose that the fundamental group of X is finite. Prove that any continuous function $f: X \to S^1$ is null-homotopic.