Topology **Comprehensive/Qualifying Exam** **August 2021**

Instructions: *Work two problems from Section A, two problems from Section B, and one problem from Section C, for a total of five problems.* Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours. Good luck!

A. **Point Set Topology** (2 problems)

A1. Let C be a subset of a topological space X.

i. Prove that if C is connected, then the closure of C is connected.

ii. Give an example where C is connected but the interior of C is not connected. (You may wish to take $X = \mathbb{R}^2$.)

A2. Let $p : X \to Y$ be a closed, continuous surjection. Prove that if Y is compact and $p^{-1}(y)$ is compact for every $y \in Y$, then X is compact. [Hint: first prove that if U is an open set containing $p^{-1}(y)$, there is a neighborhood V of y such that $p^{-1}(V) \subseteq U$.]

A3. Let $f : S^1 \to \mathbb{R}$ be continuous, where S^1 is the unit circle in $\mathbb{R}^2$.

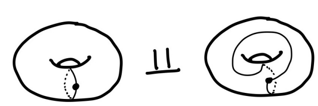
i. Show that there is a point $z \in S^1$ such that $f(z) = f(-z)$.

ii. Show that f is not surjective.

B. **Homotopy** (2 problems)

B1. Let $X = \mathbb{R}^3 \setminus \{(t, 0, 1), (t, 0, -1) \mid t \in \mathbb{R}\}$, i.e., X is Euclidean 3-space with two specific lines deleted. Compute the fundamental group of X. Justify your computation.

B2. Let $T^2 = S^1 \times S^1$ denote the torus, and let X be the quotient of $T^2 \sqcup T^2$ obtained by identifying the curves shown below. Use the Seifert–van Kampen Theorem to compute the fundamental group of X. Give your answer as a presentation.



B3. Let $X = S^1 \vee S^1$, the wedge of two circles (figure 8 space). Exhibit both a regular and an irregular 4-fold covering of X. Does X have an irregular 2-fold covering? Explain.

C. **Mixed** (1 problem)

C1. Let A be a subspace of a topological space X. A continuous map $r : X \to A$ is said to be a retraction if $r(a) = a$ for every $a \in A$.

i. Show that if $r : X \to A$ is a retraction, then the induced map $r_* : \pi_1(X, x_0) \to \pi_1(A, a_0)$ is surjective. Use this to show that there does not exist a retraction $r : D^2 \to S^1$.

ii. Prove that every continuous map $f : D^2 \to D^2$ has a fixed point.

C2. Let $p : X \to Y$ be a covering space, where X is compact, path-connected, and locally path-connected. Prove that for each $x \in X$ the set $p^{-1}(\{p(x)\})$ is finite, and has cardinality equal to the index of $p_*(\pi_1(X, x))$ in $\pi_1(Y, p(x))$.