**Instructions**: *Do* **any four** *of the following six problems. Be sure to write the number for each problem you work out and write your name clearly at the top of each page you turn in for grading. You have* $2\frac{1}{2}$ *hours.* Good luck!

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1. Consider the linear continuum $X := [0,1] \times [0,1]$ with the dictionary order.
   (a) Is $X$ connected? compact? first countable? separable? metrizable? Support each of your answers using standard theorems where appropriate.
   (b) Describe a simple basis for the subpace topology on $[0,1] \times \{0\}$.

2. Prove: $U$ is open in $X$ if and only if $\overline{U \cap \overline{A}} = \overline{U \cap A}$ for every $A \subset X$.

3. (a) Prove that a non-empty connected subset of a topological space $X$ that is both open and closed is a connected component of $X$.
   (b) Show that a connected, locally path-connected space is path-connected.

4. Let $f : X \to \mathbb{R}$ be a continuous function.
   (a) If $X$ is compact, show that $f$ has a maximum value which it attains.
   (b) if $X$ is connected and if $f(x) < c < f(z)$, show that there exists $y \in X$ such that $f(y) = c$.
   (c) If $X$ is compact and connected, show that $f(X)$ is a point or a closed interval.

5. (a) Show that a regular $T_0$ space is Hausdorff, hence $T_3$.
   (b) Show that if $f : X \to Y$ is closed, continuous, and surjective, $X$ is normal, and $Y$ is $T_1$, then $Y$ is $T_4$.

6. (a) Let $(X, d_1)$ and $(Y, d_2)$ be metric spaces. Show that
   $$d(x_1 \times y_1, x_2 \times y_2) := d_1(x_1, x_2) + d_2(y_1, y_2)$$
   defines a metric on $X \times Y$, and that the metric topology agrees with the product topology.
   (b) Suppose further that $(X, d_1)$ and $(Y, d_2)$ are complete metric spaces. Show that $(X \times Y, d)$ is a complete metric space.

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***Note: The following definitions are to be used on this exam.*** A space is

- $T_0$ if given $x \neq y$, there exists an open set $U$ that contains one of them and misses the other
- $T_1$ if singleton sets are closed sets
- $T_2$ if it is Hausdorff
- $T_3$ if it is regular and Hausdorff
- $T_4$ if it is normal and Hausdorff

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