Instructions: *Work two problems from section A, two problems from section B, and one problem from section C, for a total of five problems.* Be sure to write the number for each problem with your work, and write your name clearly at the top of each page you turn in for grading. You have three hours.

A Point Set Topology (2 problems)

A1. Let A be a subset of a topological space X. Prove that $\overline{A} - A^\circ = \overline{A} \cap \overline{X - A}$.

A2. Prove that a product of (finitely or infinitely many) connected spaces is connected.

A3. Let $f\colon S^1 \to \mathbb{R}$ be continuous, where S^1 is the unit circle in $\mathbb{R}^2$.

(a) Show that there is a point $z \in S^1$ such that $f(z) = f(-z)$.
(b) Show that f is not surjective.

B Homotopy (2 problems)

B1. (a) Prove that there is no retraction $B^2 \to S^1$.
(b) Prove that for every continuous map $f\colon B^2 \to B^2$ there is a point $x \in B^2$ with $f(x) = x$.

B2. (a) Complete the following definition: topological spaces X and Y are homotopy equivalent if ....
(b) Let X, Y and Z be topological spaces. Prove that if X is homotopy equivalent to Y and Y is homotopy equivalent to Z, then X is homotopy equivalent to Z.

B3. Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n > 2$.

C Mixed (1 problem)

C1. Let $X = Y \cup Z$, where

$$\begin{aligned} Y &= \{(0,y) \in \mathbb{R}^2 \mid -1 \le y \le 1\} \\ \text{and} \quad Z &= \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid 0 < x \le 1\}. \end{aligned}$$

Prove that X is connected but not path connected.

C2. Show that the following three conditions on a topological space X are equivalent.

(a) Every continuous map $S^1 \to X$ is null homotopic.
(b) Every continuous map $S^1 \to X$ extends to a continuous map $D^2 \to X$.
(c) The fundamental group $\pi_1(X, x_0)$ is trivial for all $x_0 \in X$.