Instructions: *Work two problems from Section A, two problems from Section B, and one problem from Section C, for a total of five problems.* Be sure to write the number for each problem you work out, and write your name clearly at the top of each page you turn in for grading. You have three hours. Good luck!

A. **Point Set Topology** (2 problems)

A1. Prove that the product of finitely many connected spaces is connected.

A2. i. Let X be a Hausdorff space and A be a compact subset of X. Prove that A is closed.

ii. Let $f : X \longrightarrow Y$ be a continuous map from a compact space X to a Hausdorff space Y. Prove that f is a closed map.

A3. Let $f : X \to Y$ be a continuous function between topological spaces X and Y, and let A be a subset of X.

i. If A is compact, prove that $f(A)$ is compact.

ii. If A is connected, prove that $f(A)$ is connected.

B. **Homotopy** (2 problems)

B1. Let X be the subspace of $\mathbb{R}^2$ that is the union of two circles of radius 1 centered at $(-2, 0)$ and $(2, 0)$ and the line segment from $(-1, 0)$ to $(1, 0)$. State the Seifert-van Kampen Theorem, and use it to find the fundamental group of X.

B2. Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.

B3. Prove the Brouwer fixed point theorem in dimensions one and two: Every continuous map $f : D^n \longrightarrow D^n$, $n = 1, 2$, has a fixed point.

C. **Mixed** (1 problem)

C1. Show that the following three conditions on a topological space X are equivalent:

i. Every continuous map $S^1 \longrightarrow X$ is null homotopic.

ii. Every continuous map $S^1 \longrightarrow X$ extends to a continuous map $D^2 \longrightarrow X$.

iii. The fundamental group $\pi_1(X, x_0)$ is trivial for all $x_0 \in X$.

C2. Show that for every finitely presented group G there is a topological space X such that the fundamental group of X is isomorphic to G.