Topology I Comprehensive Exam: January, 2001

Do any four of the following five problems.

1. (a) Let X be a set with distinct topologies $\mathcal{T}$ and $\mathcal{T}'$. Recall that two topologies on X are comparable if one is a subset of the other. Show that if $(X, \mathcal{T})$ and $(X, \mathcal{T}')$ are both compact Hausdorff, then $\mathcal{T}$ and $\mathcal{T}'$ are not comparable.

   (b) Let X be a finite set. Show that there is exactly one Hausdorff topology on X.

2. Let $\mathbf{N}$ denote the natural numbers, and consider $X = [0,1]^{\mathbf{N}}$ with the product and uniform topologies. Determine whether X with these topologies is compact, connected, or normal.

3. (a) Show that a metric space M is complete if and only if for any decreasing sequence of nonempty closed sets $\{V_n\}$ with $\operatorname{diam}(V_n) \to 0$ as $n \to \infty$, $\cap_{n=1}^{\infty} V_n$ is nonempty.

   (b) Find a decreasing sequence of nonempty subsets $\{V_n\}$ of a complete metric space with $\operatorname{diam}(V_n) \to 0$ and empty intersection.

   (c) Find a decreasing sequence of nonempty closed subsets $\{V_n\}$ of a complete metric space with empty intersection.

4. (a) Show that a compact Hausdorff space is metrizable if and only if it has a countable basis.

   (b) Let X be a locally compact metrizable Hausdorff space with one point compactification $\hat{X}$. Give an example showing that $\hat{X}$ need not be metrizable. Show that if X is second countable, then $\hat{X}$ is metrizable.

5. Let $p : X \to Y$ be a closed, continuous surjective map.

   (a) Show that if Y is connected and $p^{-1}(y)$ is connected for each $y \in Y$, then X is connected.

   (b) Show that if Y is compact and $p^{-1}(y)$ is compact for each $y \in Y$, then X is compact.