Hyperbolic Geometry and Parallel Transport in R^2_+

Tia Burden[1] Vincent Glorioso[2] Brittany Landry[3]
Phillip White[2]

[1]Department of Mathematics
Southern University
Baton Rouge, LA, USA

[2]Department of Mathematics
Southeastern Louisiana University
Hammond, LA, USA

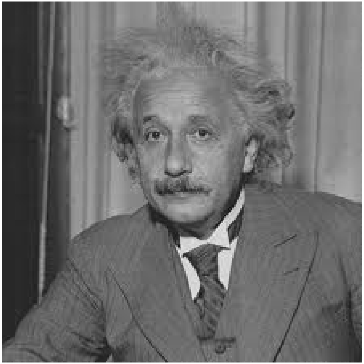
[3]Department of Mathematics
University of Alabama
Tuscaloosa, AL, USA

SMILE VIGRE Program, July, 2013

Objective

- Parallel transport along a hyperbolic triangle
- Compare angle of initial and final vector
- Compute area of hyperbolic triangle
- Compare area and angles of parallel transports of hyperbolic triangles

Albert Einstein and Hermann Minkowski

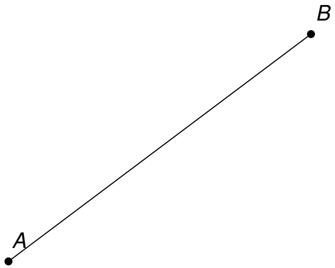


Applications

- Complex variables
- Topology of two and three dimensional manifolds
- Finitely presented infinite groups
- Physics
- Computer science

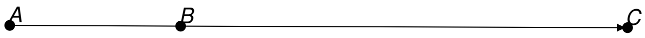
Postulate 1

A straight line segment can be drawn joining any two points.



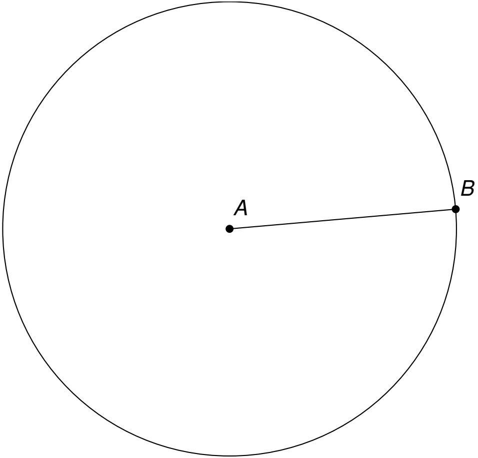
Postulate 2

Any straight line segment can be extended indefinitely in a straight line



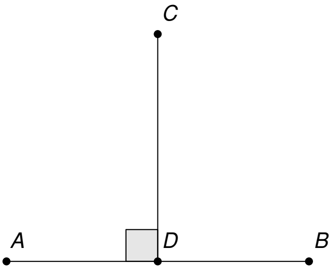
Postulate 3

Given any straight line segment, a circle can be drawn having the segment as a radius and one endpoint as center.



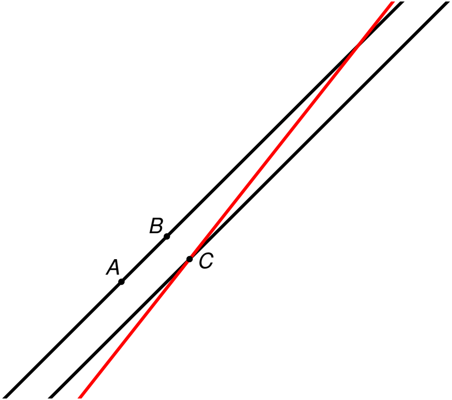
Postulate 4

All right angles are congruent.

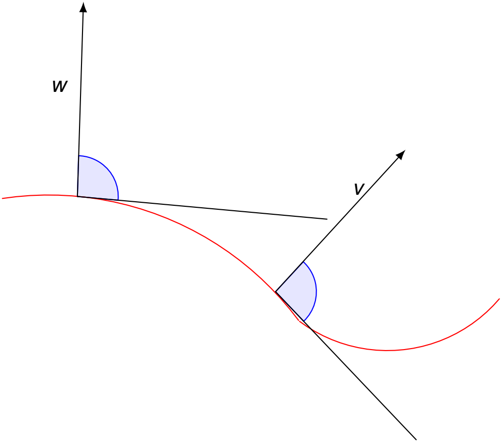


Parallel Postulate

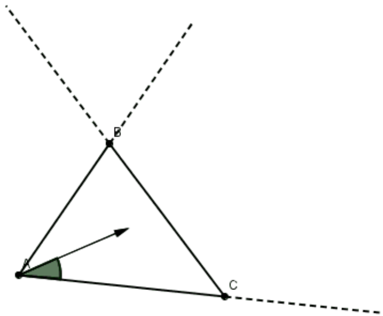
Through any given point not on a line there passes exactly one line that is parallel to that given line in the same plane.



Parallel Transport



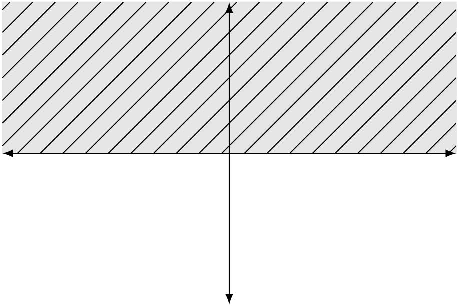
Parallel Transport About a Euclidean Triangle



Upper Half-Plane

Definition

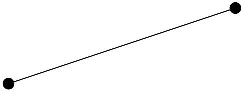
Upper half-plane: $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$.
Complex plane: $H^2 = \{x + iy : x, y \in \mathbb{R}, y > 0\}$.



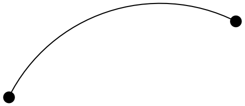
Geodesic

Definition

Geodesic is a line, curved or straight, between two points such that the acceleration of the line is 0. So given a curve γ defined on an open interval I it must satisfy $\frac{D}{dt}\frac{d\gamma}{dt} = 0 \in T_{\gamma(t)}$ for all $t \in I$.



Euclidean Geometry



Hyperbolic Geometry

Covariant Derivative

Definition

There is a unique correspondence that associates the vector field $\frac{DV}{dt}$ along the differentiable curve γ to the vector field V. The vector field V is referred to as the *covariant derivative*.

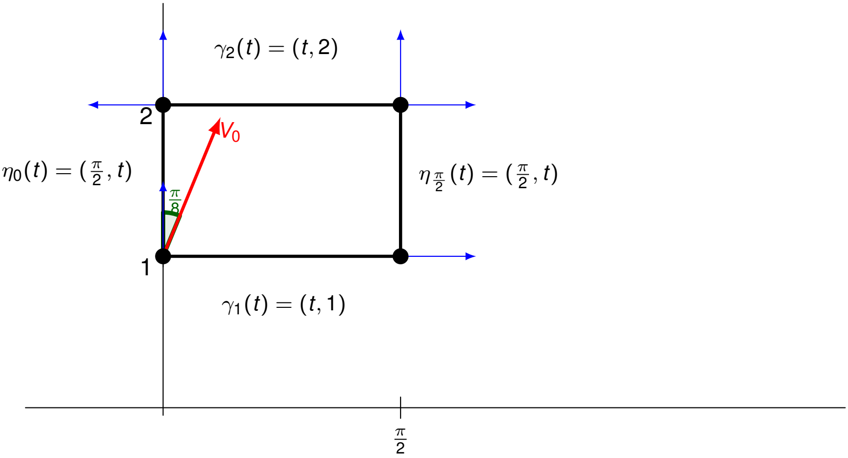
Results from the Covariant Derivative

From the Covariant Derivative we obtain a set of two differential equations, using the curve $\gamma = (\gamma_1(t), \gamma_2(t))$ and the vector field $V = (f(t), g(t))$, that a parallel vector field must satisfy.

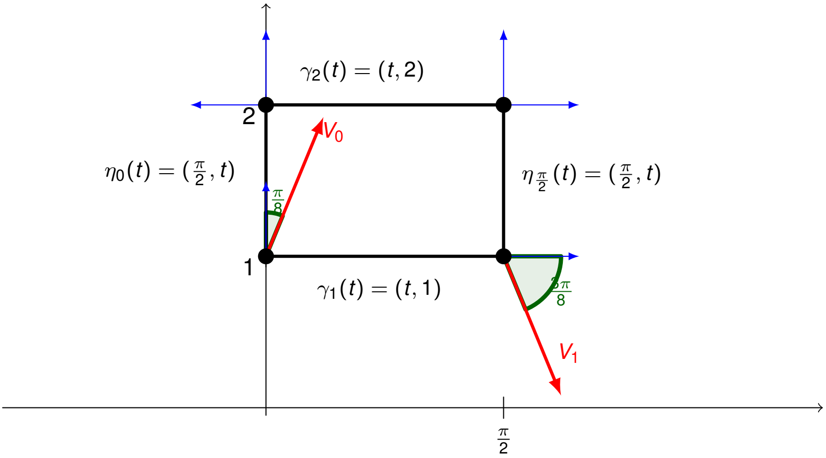
$$f'(t) = \frac{\gamma_2'(t)}{\gamma_2(t)} f(t) + \frac{\gamma_1'(t)}{\gamma_2(t)} g(t)$$

$$g'(t) = \frac{\gamma_1'(t)}{\gamma_2(t)} f(t) + \frac{\gamma_2'(t)}{\gamma_2(t)} g(t)$$

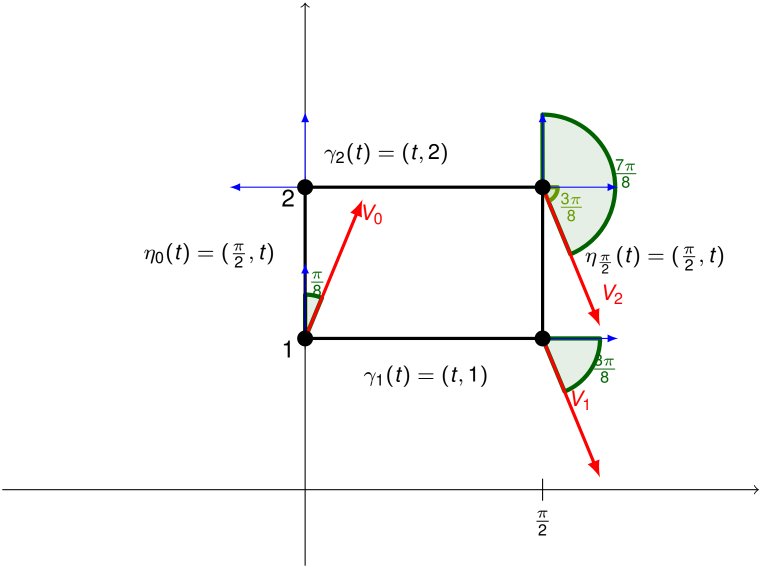
Original Vector: V_0



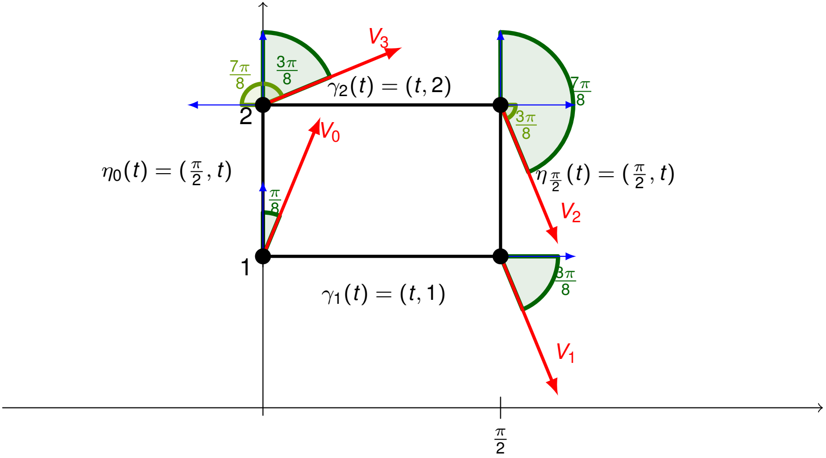
Transport Along $\gamma_1(t)$ to V_1



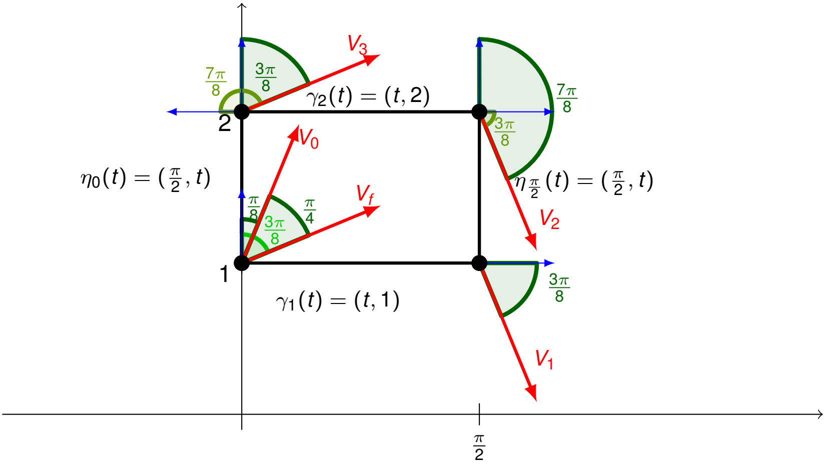
Transport Along the Geodesic $\eta_{\frac{\pi}{2}}$ to V_2



Transport Back Along $\gamma_2(t)$ to V_3



Transport Down the Geodesic η_0 to V_f



Area Calculation

To find the area of the rectangle, we use the following integral:

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} \int_1^2 \frac{dydx}{y^2} \\ &= \int_0^{\frac{\pi}{2}} (\frac{-1}{y}\Big|_1^2)\, dx \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2}\, dx \\ &= \frac{1}{2}x\Big|_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4} \end{aligned}$$

General Equation

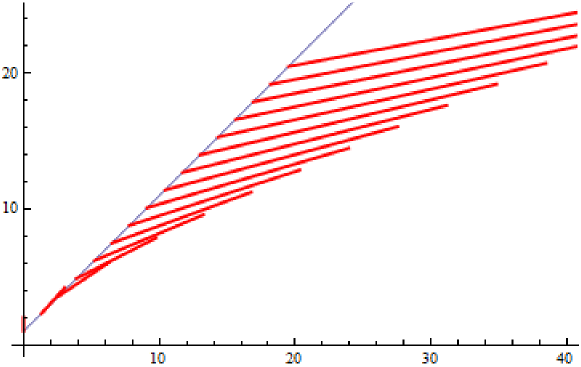
Given the curve $c(t) = (t, mt + b)$, we used Mathematica to find the equation for the parallel vector field:

$$\mathrm{V}_1(t) = e^{-\frac{t}{b+\mathrm{mt}}} \left(e^{\frac{t}{b+\mathrm{mt}}} - 1 \right),$$

$$\mathrm{V}_2(t) = e^{-\frac{t}{b+\mathrm{mt}}}$$

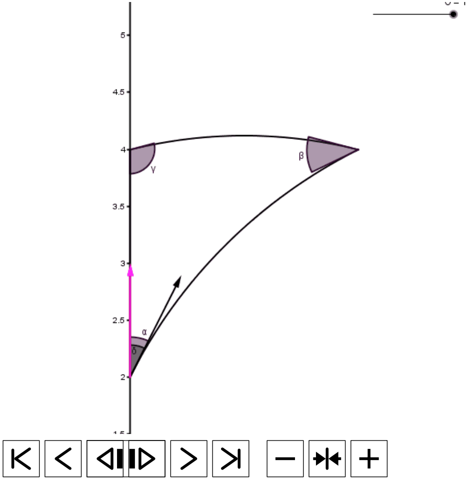
Line: $y = mx + b$

Parallel Transport about the line $2t + 1$.



General Parallel Transport

When a vector is being transported around a hyperbolic triangle it maintains the angle with the tangent vectors of the curve on which it is moving.

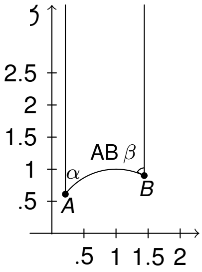


Area

Area Equation Step 1

For a hyperbolic triangle with angles α and β and $\gamma = 0$, which is an ideal triangle, we get the equation for the area:

$$A = \int_{-\cos\alpha}^{\cos\beta} \int_{\sqrt{1-x^2}}^{\infty} \frac{dydx}{y^2} = \pi - (\alpha + \beta)$$



Area Calculation

For the triangle $AB\infty$

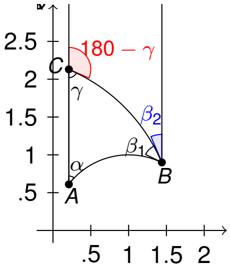
$$
\begin{aligned}
A &= \int_{-\cos\alpha}^{\cos\beta} \int_{\sqrt{1-x^2}}^{\infty} \frac{dydx}{y^2} \\
&= \int_{-\cos\alpha}^{\cos\beta} \frac{-1}{y}\Big|_{\sqrt{1-x^2}}^{\infty} dx \\
&= \int_{-\cos\alpha}^{\cos\beta} \frac{1}{\sqrt{1-x^2}} dx \\
&= \arcsin(x)\Big|_{-\cos\alpha}^{\cos\beta} \\
&= \arcsin(\sin(\frac{\pi}{2}-\beta)) - \arcsin(-\sin(\frac{\pi}{2}-\alpha)) \\
&= ((\frac{\pi}{2}-\beta) + (\frac{\pi}{2}-\alpha)) \\
&= \pi - (\alpha+\beta)
\end{aligned}
$$

Area

Area equation

We can find the area of a general hyperbolic triangle with angles α, β, and γ by subtracting the areas of two ideal hyperbolic triangles:

$$A = \pi - (\alpha + \beta + \gamma)$$



Area Calculation

For the triangle *ABC*

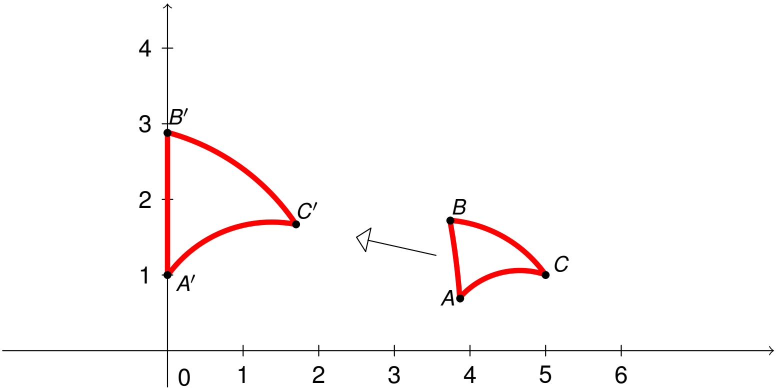
$$
\begin{aligned}
A &= \int_{-\cos\alpha}^{\cos\beta}\int_{\sqrt{1-x^2}}^{\infty}\frac{dydx}{y^2} - \int_{-\cos(\pi-\gamma)}^{\cos\beta_2}\int_{\sqrt{1-x^2}}^{\infty}\frac{dydx}{y^2} \\
&= \int_{-\cos\alpha}^{\cos\beta}\frac{-1}{y}\Big|_{\sqrt{1-x^2}}^{\infty}dx - \int_{-\cos(\pi-\gamma)}^{\cos\beta_2}\frac{-1}{y}\Big|_{\sqrt{1-x^2}}^{\infty}dx \\
&= \int_{-\cos\alpha}^{\cos\beta}\frac{1}{\sqrt{1-x^2}}dx - \int_{-\cos(\pi-\gamma)}^{\cos\beta_2}\frac{1}{\sqrt{1-x^2}}dx \\
&= \arcsin(x)\Big|_{-\cos\alpha}^{\cos\beta} - \arcsin(x)\Big|_{-\cos(\pi-\gamma)}^{\cos\beta_2} \\
&= ((\frac{\pi}{2}-\beta)+(\frac{\pi}{2}-\alpha)) - ((\frac{\pi}{2}-\beta_2)+(\frac{\pi}{2}-\gamma)) \\
&= \pi-(\alpha+\beta)-(\pi-(\gamma+\beta_2)) \\
&= \pi-(\alpha+\beta_1+\beta_2)-(\pi-(\gamma+\beta_2)) \\
&= \pi-(\alpha+\beta+\gamma)
\end{aligned}
$$

Area vs. Angle of Initial to Transported Vector

Theorem

The area of any hyperbolic triangle in $\mathbb{R}^2_+$ is equal to the angle between the initial and final transported vectors.

Non-Normal Hyperbolic Triangle



What are Fractional Transformations?

A fractional transformation is a special representation of a matrix that is used in Möbius Transformations.

Fractional Transformation

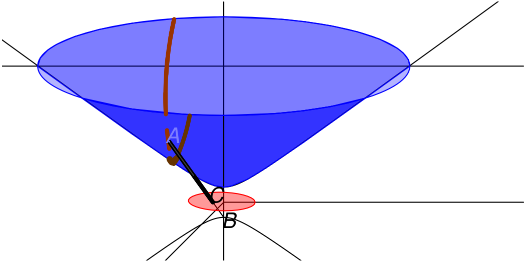
Given matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \in \mathbb{R}$, and the point $p = (x, y)$. The fractional transformation of p in terms of M is written as:

$$f_M(p) = \frac{ap + b}{cp + d} = \frac{a(x + yi) + b}{c(x + yi) + d}$$

Future Studies

Parallel Transport in Different Models of Hyperbolic Geometry

- Poincaré Disk
- Hyperboloid



Acknowledgments

We would like to give a big thanks to our graduate student Andrew Holmes for helping us with this project, in addition to Dr. Edgar Reyes for providing the information needed for this project. Also, we would like to thank the SMILE program for allowing us to have this great opportunity.