CALCULATING A^k USING FULMER'S METHOD

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ABSTRACT. In [4], A^k is computed using partial fraction decomposition, which can be computationally intense. In this paper we developed a simpler method for any $n \times n$ matrix. A argument will be made showing that Fulmer's Method can be used to compute A^k, and extended to our results to compute e^{At}. Lastly, we will formulate a formula to compute the matrix exponential.

1. Introduction

In real life, one may find that representing data in matrix form is a compact and easy way to visualize data. Matrices may also be used to solve difference equations, where the answer is dependent on a square matrix raised to a power. Matrices are also used to represent system of differential equations in a compact way, using capital letters to represent the matrices. Bearing the structure on a first order linear differential equation, one may question whether or not we may use the same method to solve a matrix first order linear differential equation. The answer to that question is that we can, provided that we may calculate the matrix exponential. From Calculus, we find that using the power series for e^x, we may define the matrix exponential as

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

where A is a $n \times n$ matrix. Although this formula is correct and valid, one can also wonder if there is an easier way to calculate e^{At} instead of adding up an infinite amount of terms, which includes a matrix to a potential high power.

2. Preliminaries

Before we start to calculate a formula for raising a square matrix to a power, we must review some linear algebra terms and define an operator. These theorems and definitions will come from [2] and [1].

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Definition 2.1. Let V be a vector space and S be a nonempty subset of V. A vector $v \in V$ is called a **linear combination** of vectors of S if there exists a finite number of vectors $u_1, u_2, \cdots, u_n$ in S and scalars $a_1, a_2, \cdots, a_n$ in F such that $v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$. In this case we say that v is a linear combination of $u_1, u_2, \cdots, u_n$ and call $a_1, a_2, \cdots, a_n$ the **coefficients** of the linear combination.

Definition 2.2. A set of vectors $v_1, v_2, .., v_n$ are **linearly independent** if whenever $a_1v_1 + a_2v_2 + ... + a_nv_n = 0$, then $a_1 = a_2 = ... = a_n = 0$.

Definition 2.3. A **basis** β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β for a basis for V.

Theorem 2.4. *Let V be a vector space and $\beta = \{u_1, u_2, \cdots, u_n\}$ be a subset of V. Then β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β, that is, can be expressed*

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for unique scalars $a_1, a_2, \cdots, a_n$.

Definition 2.5. Let $a(k)$ be a sequence of complex numbers and n is an integer. We define $E\{a(k)\}$ as a shift operator for sequences where

$$E^n\{a(k)\} = a(k+n)$$

Lemma 2.6. *The E shift operator is a linear operator.*

Proof. Let $a(k)$ and $b(k)$ be sequences with complex terms and c be a complex constant.

(1)

$$\begin{aligned} E^n\{a(k) + b(k)\} &= a(k+n) + b(k+n) \\ &= E^n\{a(k)\} + E^n\{b(k)\} \end{aligned}$$

(2)

$$\begin{aligned} E^n\{ca(k)\} &= ca(k+n) \\ &= cE^n\{a(k)\} \end{aligned}$$

□

Definition 2.7. Let n be a non-negative integer. The **falling factorial** is the sequence $k^{\underline{n}}$, with $k = 0, 1, 2, \ldots$ given by the following formula.

$$k^{\underline{n}} = k(k-1)(k-2)\cdots(k-n+1).$$

If k were allowed to be a real variable then $k^{\underline{n}}$ could be characterized as the unique monic polynomial of degree n that vanishes at $0, 1, \ldots, n-1$. Observe also that $k^{\underline{n}} j_{k=n} = n!$.

Definition 2.8. Let n be a non-negative integer and a be a complex number. We define a sequence $\varphi_{n,a}(k)$ as

$$\varphi_{n,a}(k) = \begin{cases} \frac{a^{k-n} k^{\underline{n}}}{n!} & a \neq 0, \\ \delta_n(k) & a = 0. \end{cases}$$

where

$$\delta_n(k) = \begin{cases} 0 & k \neq n \\ 1 & k = n \end{cases}$$

Example 2.9. With $\varphi_{0,0}(k), \varphi_{1,4}(k)$, and $\varphi_{2,5}(k)$, we have

$$\begin{array}{lll} \varphi_{0,0}(k) & = \delta_0(k) & = (1, 0, 0, 0, 0, \ldots) \\ \varphi_{1,4}(k) & = 4^{k-1}k & = (0, 1, 8, 48, 256, \ldots) \\ \varphi_{2,5}(k) & = \frac{5^{k-2}k(k-1)}{2} & = (0, 0, 1, 15, 150, \ldots) \end{array}$$

Although seemingly different, these two cases are related to one another. We cannot allow $a = 0$ in the case that we may have negative powers ($k < n$). However, if $k < n$, then $k^{\underline{n}}$ is zero. Therefore, we see that

$$\lim_{a \to 0} \frac{a^{k-n} k^{\underline{n}}}{n!} = \delta_n(k)$$

where the limit is understood in a point-wise sense.

Lemma 2.10. *Let $\boldsymbol{D}$ denote the ordinary derivative operator. Let n be a non-negative integer and $a \in \mathbb{C}$. We then have*

$$\varphi_{n,a}(k) = \left. \frac{\boldsymbol{D}^n x^k}{n!} \right|_{x=a}$$

where the notation $|_{x=a}$, is to be understood in the limit sense.

Proof.

$$\begin{aligned} \frac{\mathbf{D}^n(x^k)}{n!} &= \frac{k(k-1)(k-2)(k-3)...(k-n+1)x^{k-n}}{n!} \\ &= \frac{k^{\underline{n}} x^{k-n}}{n!} \end{aligned}$$

Evaluating at $x = a$ in the case a $\neq 0$ gives $\varphi_{n,0}(k)$. If $a = 0$ then as explained above, we get

$$\lim_{x \to 0} \frac{k^{\underline{n}} x^{k-n}}{n!} = \varphi_n(k) = \varphi_{n,0}(k)$$

□

3. The $\mathcal{Z}$-transform

With our previous definition, we may now define the $\mathcal{Z}$-transform and its various properties which are essential to calculating A^k. The following definitions and theorems were obtain from [4].

Definition 3.1. Let $y(k)$ be a sequence of complex numbers. We define the **$\mathcal{Z}$-transform** of $y(k)$ to be the function $\mathcal{Z}\{y(k)\}(z)$, where z is a complex variable, by the following formula:

$$\mathcal{Z}\{y(k)\}(z) = \sum_{k=0}^{\infty} \frac{y(k)}{z^k}$$

With this definition of the $\mathcal{Z}$-transform, we find a set of properties that arises from it.

Proposition 3.2. *Suppose a is a nonzero complex number, $n \in \mathbb{C}$ and $n \in \mathbb{N}$ = 0,1,2,..., and y(k) is a sequence for which the $\mathcal{Z}$ -transform exists.*
Then
(1) $\mathcal{Z}\{a^k\}(z) = \frac{z}{z-a}$
(2) $\mathcal{Z}\{a^k y(k)\}(z) = Y(\frac{z}{a})$
(3) $\mathcal{Z}\{y(k+n)\}(z) = z^n Y(z) - \sum_{m=0}^{n-1} y(m) z^{n-m}$
(4) $\mathcal{Z}\{(k+n-1)^{\underline{n}} y(k)\}(z) = (-1)^n z^n \mathbf{D}^n Y(z)$
(5) $\mathcal{Z}\{k^{\underline{n}}\}(z) = \frac{n!z}{(z-1)^{n+1}}$

Proof.
(1) $\mathcal{Z}\{a^k\}(z) = \sum_{k=0}^{\infty} \frac{a^k}{z^k} = \sum_{k=0}^{\infty} (\frac{a}{z})^k = \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}$
(2) $\mathcal{Z}\{a^k y(k)\}(z) = \sum_{k=1}^{\infty} \frac{a^k y(k)}{z^k} = \sum_{k=1}^{\infty} \frac{y(k)}{z/a^k} = Y(\frac{z}{a})$
(3)

$$\begin{aligned}\mathcal{Z}\{y(k+n)\}(z) &= \sum_{k=0}^{\infty} \frac{y(k+n)}{z^k} = z^n \sum_{k=n}^{\infty} \frac{y(k)}{z^k} \\ &= z^n (\sum_{k=0}^{\infty} \frac{y(k)}{z^k} - \sum_{m=0}^{n-1} \frac{y(m)}{z^m}) \\ &= z^n Y(z) - \sum_{m=0}^{n-1} y(m) z^{n-m}\end{aligned}$$

(4)

$$\begin{aligned}
\frac{d^nY(z)}{dz^n} &= (-1)^n \sum_{k=0}^{\infty} k(k+1)...(k+n-1)y(k)z^{-k-n} \\
&= \frac{(-1)^n}{z^n} \sum_{k=0}^{\infty} \frac{(k+n-1)^{\underline{n}}y(k)}{z^k} \\
&= \frac{(-1)^n}{z^n} \mathcal{Z}\{(k+n-1)^{\underline{n}}y(k)\}(z)
\end{aligned}$$

(5) Let y(k) = $1^k = 1$. Then Y(z) = $\mathcal{Z}\{1\}$ (z) =

$$\begin{aligned}
\frac{(-1)^n n!}{(z-1)^{n+1}} &= \frac{(-1)^n}{z^n}\mathcal{Z}\{(k+n+-1)^{\underline{n}}\} \\
&= \frac{(-1)^n}{z^n}(z^{n-1}\mathcal{Z}\{k^{\underline{n}}\}(z) - \sum_{m=0}^{n-2} m^{\underline{n}} z^{n-m-1}) \\
&= \frac{(-1)^n}{z}\mathcal{Z}\{k^{\underline{n}}\}(z)
\end{aligned}$$

□

From these 5 $\mathcal{Z}$-transforms, we may prove the more complex $\mathcal{Z}$-transform that will lead to a formula for raising a matrix to a power.

Proposition 3.3. *Let $a \in \mathbb{C}$ and $n \in \mathbb{N}$. With $\varphi_{n,a}$ given in Definition we have*

$$\mathcal{Z}\{\varphi_{n,a}(k)\}(z) = \frac{z}{(z-a)^{n+1}}$$

Proof. First assume $a \neq 0$. Then $\varphi_{n,a}(k) = \frac{a^{k-n}k^{\underline{n}}}{n!} = \frac{a^{-n}}{n!}a^k k^{\underline{n}}$. We use formula (2) and (5) in the previous proposition to get

$$\begin{aligned}
\mathcal{Z}\{\varphi_{n,a}(k)\}(z) &= \frac{a^{-n}}{n!}\mathcal{Z}\{a^k k^{\underline{n}}\}(z) \\
&= \frac{a^{-n}}{n!}\mathcal{Z}\{k^{\underline{n}}\}(z/a) \\
&= \frac{a^{-n}}{n!}\frac{n!(z/a)}{(\frac{z}{a}-1)^{n+1}} \\
&= \frac{z}{(z-a)^{n+1}}
\end{aligned}$$

Now Suppose $a = 0$. Then $\varphi_{n,0}(k) = \delta_n(k)$. From the definition of the $\mathcal{Z}$-transform we get

$$\mathcal{Z}\{\varphi_{n,0}(k)\}(z) = \sum_{k=0}^{\infty} \frac{\delta_n(k)}{z^k}$$

However, since every single term except the n^{th} term is 0, we only have

$$\begin{aligned}\mathcal{Z}\{\varphi_{n,0}(k)\}(z) &= \sum_{k=0}^{\infty} \frac{\delta_n(k)}{z^k} \\ &= \frac{1}{z^n} \\ &= \frac{z}{(z-0)^{n+1}}\end{aligned}$$

No matter what value of a we use, we will end up with the same formula. □

Suppose $\mathbf{y}(k)$ is a sequence of $n \times n$ matrices over $\mathbb{C}$. We can extend the $\mathcal{Z}$-transform to $\mathbf{y}(k)$ by applying it to each entry. Equation 3 of Proposition 3.2 extends to this matrix valued case; the proof is verbatim the same.

Let A be a $n \times n$ matrix. Our next proposition in a description of the $\mathcal{Z}$-Transform of the sequence of the matrices A^k.

Proposition 3.4. *Let A be an $n \times n$ matrix with entries in the complex plane. Then*

$$\mathcal{Z}\{A^k\}(z) = z(zI - A)^{-1}$$

where I is the $n \times n$ identity matrix.

Proof. Let $y(k) = A^k$, letting $y(0) = I$. Then by induction, $y(k+1) = Ay(k)$. Applying the $\mathcal{Z}$-Transform to both sides and using Proposition 3.2 yields

$$z\mathcal{Z}\{A^k\}(z) - zI = A\mathcal{Z}\{A^k\}(z)$$

Solving for $\mathcal{Z}\{A^k\}(z)$ will yield the above proposition. □

With this proposition, we would finally have a formula for A^k should the $\mathcal{Z}$-Transform have an inverse and as it turns out, it does.

Proposition 3.5. *The $\mathcal{Z}$-Transform is linear and one-to-one on the set of sequences for which the $\mathcal{Z}$-Transform exists.*

Proof. Let $a(k)$ and $b(k)$ be both sequences and c be a complex number. Then we have

$$\begin{aligned}
\mathcal{Z}\{a(k)+b(k)\}(z) &= \sum_{k=0}^{\infty} \frac{a(k)+b(k)}{z^k} \\
&= \sum_{k=0}^{\infty} \frac{a(k)}{z^k} + \sum_{k=0}^{\infty} \frac{b(k)}{z^k} \\
&= \mathcal{Z}\{a(k)\}(z) + \mathcal{Z}\{b(k)\}(z) \\
\mathcal{Z}\{ca(k)\}(z) &= \sum_{k=0}^{\infty} \frac{ca(k)}{z^k} \\
&= c\sum_{k=0}^{\infty} \frac{a(k)}{z^k} \\
&= c\mathcal{Z}\{a(k)\}(z)
\end{aligned}$$

Therefore, the $\mathcal{Z}$-Transform is a linear transformation. Now suppose that the $\mathcal{Z}$-Transform for $a(k)$ and $b(k)$ exists and that they equal each other. Letting $w = z^{-1}$, we have

$$\sum_{k=0}^{\infty} a(k)w^k = \sum_{k=0}^{\infty} b(k)w^k$$

Taking the n^{th} derivative of both sides and evaluating at $w = 0$, we have that

$$n!a(n) = n!b(n)$$

Since $n!$ is a constant, we can divide it out and come to the conclusion that $a(k) = b(k)$ for all k. Therefore, the $\mathcal{Z}$-Transform is one-to-one. □

Because that the $\mathcal{Z}$-Transform is one-to-one, we may also define the **inverse $\mathcal{Z}$-Transform** as

$$\mathcal{Z}^{-1}\{A(z)\}(k) = a(k)$$

where $A(z)$ is

$$\sum_{k=0}^{\infty} \frac{a(k)}{z^k}$$

4. Results

Because of Proposition 3.5, we finally have a formula for A^k. Taking the inverse $\mathcal{Z}$-Transform of both sides, we find that

$$A^k = \mathcal{Z}^{-1}\{z(zI-A)^{-1}\}$$

However, we will see that directly calculating it might be a bit cumbersome.

Example 4.1. Find A^k if

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

First, we would compute $zI - A$.

$$zI - A = \begin{bmatrix} z-2 & 1 \\ -1 & z \end{bmatrix}$$

Next, compute the inverse of $zI - A$.

$$\begin{aligned} (zI - A)^{-1} &= \frac{1}{z^2 - 2z + 1} \begin{bmatrix} z & -1 \\ 1 & z-2 \end{bmatrix} \\ &= \frac{1}{(z-1)^2} \begin{bmatrix} z & -1 \\ 1 & z-2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{z}{(z-1)^2} & \frac{-1}{(z-1)^2} \\ \frac{1}{(z-1)^2} & \frac{z-2}{(z-1)^2} \end{bmatrix} \end{aligned}$$

However, even when we multiply z into the matrix, we find that we cannot take the inverse $\mathcal{Z}$-transform of the matrix. So we must perform partial fraction decomposition to obtain

$$\frac{z}{(z-1)^2} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2}$$

$$z = A(z-1) + B$$

if $z = 1$

$$\begin{aligned} 1 &= A(1-1) + B \\ 1 &= A(0) + B \\ 1 &= 0 + B \\ B &= 1 \end{aligned}$$

if $z = 0$

$$\begin{aligned} 0 &= A(0-1) + B \\ 0 &= A(-1) + B \\ 0 &= -A + B \\ A &= B \\ A &= 1 \end{aligned}$$

Plug in results:

$$\frac{z}{(z-1)^2} = \frac{1}{(z-1)} + \frac{1}{(z-1)^2}$$

$$\frac{z-2}{(z-1)^2} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2}$$

$$z-2 = A(z-1) + B$$

if $z = 1$

$$\begin{aligned} 1-2 &= A(1-1) + B \\ -1 &= A(0) + B \\ -1 &= 0 + B \\ B &= -1 \end{aligned}$$

if $z = 0$

$$\begin{aligned} 0-2 &= A(0-1) + B \\ -2 &= A(-1) + B \\ -2 &= -A + B \\ -2 &= -A + B \\ 2 + B &= A \\ 2 + (-1) &= A \\ 1 &= A \end{aligned}$$

Plug in results:

$$\frac{z-2}{(z-1)^2} = \frac{1}{(z-1)} + \frac{-1}{(z-1)^2}$$

$$(zI-A)^{-1} = \begin{bmatrix} \frac{1}{(z-1)} + \frac{1}{(z-1)^2} & \frac{-1}{(z-1)^2} \\ \frac{1}{(z-1)^2} & \frac{1}{(z-1)} + \frac{-1}{(z-1)^2} \end{bmatrix}$$

$$(zI-A)^{-1} = \frac{1}{z-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{(z-1)^2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

separating the original matrix into two matrices that have a common denominator for each entry.

Multiplying z into the equation, we obtain

$$z(zI-A)^{-1} = \frac{z}{z-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{z}{(z-1)^2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

From Proposition 3.3 and taking the inverse $\mathcal{Z}$-transform, we find that

$$\varphi_{n,a}(k) = \mathcal{Z}^{-1}\left\{\frac{z}{(z-a)^{n+1}}\right\}$$

and since we have fractions that are of this form, we may apply the inverse Z-transform to obtain

$$\begin{aligned} A^k &= Z^{-1}\left\{\frac{z}{z-1}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{z}{(z-1)^2}\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}\right\} \\ &= \varphi_{0,1}(k)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \varphi_{1,1}(k)\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix} \\ &= \frac{1^{k-0}k^{\underline{0}}}{0!}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{1^{k-1}k^{\underline{1}}}{1!}\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix} \\ &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + k\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix} \end{aligned}$$

Therefore, we obtain that

$$A^k = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + k\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}$$

For calculating A^k, it seems that we may have to

(1) Compute $(zI - A)^{-1}$
(2) Perform partial fraction decomposition to be able perform the inverse $\mathcal{Z}$-transform.
(3) Apply the inverse $\mathcal{Z}$-transform.

However, we may obtain a variation of Fulmer's method to ease the computations and obtaining an alternate formula for A^k. From [4], we see that powers of a matrix A are calculated by

$$A^k = \sum_{a=1}^{r}\sum_{n=0}^{m_a-1} M_{na}\ \varphi_{n,a_r}(k)$$

where a_r is a distinct eigenvalue of matrix A, m_a is the multiplicity of a_r and M_{na} the coefficient matrix associated to the phi sequence. We can easily find the phi sequences but is it always a guarantee that we may find the constant matrices that are associated with those phi sequences? The next two theorems will answer this question and will also form the basis for the variation of Fulmer's method.

Theorem 4.2. *The standard basis* $\mathbb{B}_q = \{\varphi_{1\lambda_1}, \varphi_{1\lambda_2}, \cdots, \varphi_{2\lambda_1}, \varphi_{2\lambda_2}, \cdots \varphi_{r\lambda_m}\}$ *is linearly independent.*

Proof. Write $\mathbb{B}_q$ as a linear combination.

$$\begin{aligned}
&a_{11}\varphi_{1\lambda_1} + a_{12}\varphi_{1\lambda_2} + \cdots + a_{1m}\varphi_{1\lambda_m} + \\
&a_{21}\varphi_{2\lambda_1} + a_{22}\varphi_{2\lambda_2} + \cdots + a_{2m}\varphi_{2\lambda_m} + \\
&\vdots \\
&a_{r1}\varphi_{r\lambda_1} + a_{r2}\varphi_{r\lambda_2} + \cdots + a_{rm}\varphi_{r\lambda_m} = 0
\end{aligned}$$

Apply the $\mathcal{Z}$-Transform and using Proposition 3.5 and Proposition 3.3, we find that

$$\begin{aligned}
&a_{11}\frac{z}{(z-\lambda_1)^2} + a_{12}\frac{z}{(z-\lambda_2)^2} + \cdots + a_{1m}\frac{z}{(z-\lambda_m)^2} + \\
&a_{21}\frac{z}{(z-\lambda_1)^3} + a_{22}\frac{z}{(z-\lambda_2)^3} + \cdots + a_{2m}\frac{z}{(z-\lambda_m)^3} + \\
&\vdots \\
&a_{r1}\frac{z}{(z-\lambda_1)^{r+1}} + a_{r2}\frac{z}{(z-\lambda_2)^{r+1}} + \cdots + a_{rm}\frac{z}{(z-\lambda_m)^{r+1}} = 0
\end{aligned}$$

Regroup to form like denominators.

$$\begin{aligned}
&a_{11}\frac{z}{(z-\lambda_1)^2} + a_{21}\frac{z}{(z-\lambda_1)^3} + \cdots + a_{r1}\frac{z}{(z-1)^{r+\lambda_1}} + \\
&a_{12}\frac{z}{(z-\lambda_2)^2} + a_{22}\frac{z}{(z-\lambda_2)^3} + \cdots + a_{r2}\frac{z}{(z-\lambda_2)^{r+1}} + \\
&\vdots \\
&a_{1m}\frac{z}{(z-\lambda_m)^2} + a_{2m}\frac{z}{(z-\lambda_m)^3} + \cdots + a_{rm}\frac{z}{(z-\lambda_m)^{r+1}} = 0
\end{aligned}$$

Multiply the top and the bottom to get polynomials.

$$\begin{aligned}
&a_{11}\frac{z(z-\lambda_1)^{r-1}}{(z-\lambda_1)^{r+1}} + a_{21}\frac{z(z-\lambda_1)^{r-2}}{(z-\lambda_1)^{r+1}} + \cdots + a_{r1}\frac{z}{(z-\lambda_1)^{r+1}} + \\
&a_{12}\frac{z(z-\lambda_2)^{r-1}}{(z-\lambda_2)^{r+1}} + a_{22}\frac{z(z-\lambda_2)^{r-2}}{(z-\lambda_2)^{r+1}} + \cdots + a_{r2}\frac{z}{(z-\lambda_2)^{r+1}} + \\
&\vdots \\
&a_{1m}\frac{z(z-\lambda_m)^{r-1}}{(z-\lambda_m)^{r+1}} + a_{2m}\frac{z(z-\lambda_m)^{r-2}}{(z-\lambda_m)^{r+1}} + \cdots + a_{rm}\frac{z}{(z-\lambda_m)^{r+1}} = 0
\end{aligned}$$

Simplify

$$\frac{a_{11}(z(z-\lambda_1)^{r-1})+a_{21}(z(z-\lambda_1)^{r-2})+\cdots+a_{r1}(z)}{(z-\lambda_1)^{r+1}}+$$
$$\frac{a_{12}(z(z-\lambda_2)^{r-1})+a_{22}(z(z-\lambda_2)^{r-2})+\cdots+a_{r2}(z)}{(z-\lambda_2)^{r+1}}+$$
$$\vdots$$
$$\frac{a_{1m}(z(z-\lambda_m)^{r-1})+a_{2m}(z(z-\lambda_m)^{r-2})+\cdots+a_{rm}(z)}{(z-\lambda_m)^{r+1}}=0$$

Let $n_b(z-\lambda_b)$ be a polynomial. So we get

$$\frac{n_1(z-\lambda_1)}{(z-1)^{r+1}}+\frac{n_2(z-\lambda_2)}{(z-\lambda_2)^{r+1}}+\cdots+\frac{n_m(z-\lambda_m)}{(z-\lambda_m)^{r+1}}=0$$

If $n_1(z-\lambda_1)\neq 0$, then

$$\lim_{z\to\lambda_1}\left[\frac{n_1(z-\lambda_1)}{(z-\lambda_1)^{r+1}}+\frac{n_2(z-\lambda_2)}{(z-\lambda_2)^{r+1}}+\cdots+\frac{n_m(z-\lambda_m)}{(z-\lambda_m)^{r+1}}\right]=\infty+C=0$$

where C is a constant. Thus, we get a contradiction. Therefore, $n_1(z-\lambda_1)=0$ which implies that $n_1(z)=0$ and $a_{11},a_{12},\cdots,a_{1m}=0$. You can continue this argument by induction to obtain $\forall a's=0$ □

With Theorem 4.2, we may now show that we may apply Fulmer's method to any $n\times n$ matrix.

Theorem 4.3. *When calculating A^k, we find that the set of coefficient matrices are unique to a square matrix A*

Proof. We know that

$$A^k=\sum_{a=1}^{r}\sum_{n=0}^{m_a-1}M_{na}\;\varphi_{n,a_r}(k)$$

where a_r is a distinct eigenvalue, and m_a is the multiplicity of a_r. However, since there are a finite number of terms, we may drop the double subscript in favor of a single subscript.

$$A^k=\sum_{n=1}^{R}M_n\;\varphi_n(k)$$

If we know all of the $\varphi_n(k)$ sequences and the coefficient matrices are the unknown, we may create R system of equations using the E operator.

Setting k = 0, we have

$$\begin{aligned}
A^k \quad &= I &&= M_1\varphi_1(0) + M_2\varphi_2(0) + M_3\varphi_3(0) + \dots \\
E\{A^k\} \quad &= A &&= M_1\varphi_1(1) + M_2\varphi_2(1) + M_3\varphi_3(1) + \dots \\
E^2\{A^k\} \quad &= A^2 &&= M_1\varphi_1(2) + M_2\varphi_2(2) + M_3\varphi_3(2) + \dots \\
\vdots \quad &= \vdots &&= \vdots \\
E^{R-1}\{A^k\} \quad &= A^{R-1} &&= M_1\varphi_1(R-1) + M_2\varphi_2(R-1) + M_3\varphi_3(R-1) + \dots
\end{aligned}$$

We may also represent this system of equations as a matrix equation.

$$\begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^{R-1} \end{bmatrix} = \begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \dots & \varphi_R(0) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \dots & \varphi_R(1) \\ \varphi_1(2) & \varphi_2(2) & \varphi_3(2) & \dots & \varphi_R(2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_1(R-1) & \varphi_2(R-1) & \varphi_3(3) & \dots & \varphi_R(R-1) \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_R \end{bmatrix}$$

We will let B be equal to

$$\begin{bmatrix} \varphi_1(k) & \varphi_2(k) & \varphi_3(k) & \dots & \varphi_R(k) \\ \varphi_1(k+1) & \varphi_2(k+1) & \varphi_3(k+1) & \dots & \varphi_R(k+1) \\ \varphi_1(k+2) & \varphi_2(k+2) & \varphi_3(k+2) & \dots & \varphi_R(k+2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_1(k+R-1) & \varphi_2(k+R-1) & \varphi_3(k+3) & \dots & \varphi_R(k+R-1) \end{bmatrix}$$

The following results can be found in [3]. The matrix equation has a unique solution if and only if B has nonzero determinant. B is actually a matrix called the **matrix of Casorati** and its determinant is called the Casoratian. A particular theorem that arises from **The Difference equation** states that if the set of functions is linear independent, then their Casoratian is nonzero. Because the functions that we have obtained are linearly independent from Theorem 4.2, then the determinant is nonzero. Therefore, the solution obtained is a unique solution and we may apply this to any number of system of equations. Therefore, we may write powers of any square matrix and calculate the coefficient matrices. □

So with Theorem 4.2 and Theorem 4.3 we may create a variation of Fulmer's method to calculate A^k where we would have to

(1) Calculate the eigenvalues of A, taking note of the multiplicities of each eigenvalue.
(2) Associate the correct phi sequences, accounting for all eigenvalues and their respective multiplicities.

(3) Create a system of equations from using the E shift operator $n-1$ times and evaluate at $k=0$.
(4) Finally, we solve for the coefficient matrices.

Using this method, we circumvent the need to calculate the inverse of $(zI-A)$ and the usage of partial fraction decomposition for the inverse $\mathcal{Z}$-Transform in favor of solving a system of equations.

Example 4.4. Find A^k by using the Fulmer's Method if

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

Recall $A^k = \mathcal{Z}^{-1}\{z(zI-A)^{-1}\}$. We first want to compute the determinant of $z(zI-A)^{-1}$. We start by computing $zI-A$,

$$zI - A = \begin{bmatrix} z-2 & 1 \\ -1 & z \end{bmatrix}.$$

Next,

$$\det(zI-A) = (z-1)^2$$

this implies

$$\det((zI-A)^{-1}) = \frac{1}{(z-1)^2}.$$

Lastly, we must multiply both sides by z

$$z\det((zI-A)^{-1}) = \frac{z}{(z-1)^2}.$$

Since $z\det((zI-A)^{-1}) = \frac{z}{(z-1)^2}$ we apply the inverse $\mathcal{Z}$-transform, and we obtain that A^k can be written as a linear combination of $\{\varphi_{0,1}, \varphi_{1,1}\}$.

We can write A^k in the following way:

$$A^k = \varphi_{0,1}(k)M + \varphi_{1,1}(k)N,$$

where M and N are our unknown matrices. By the way we defined the ϕ function we obtain

$$\begin{aligned} A^k &= \frac{1^{k-0}k^{\underline{0}}}{0!}M + \frac{1^{k-1}k^{\underline{1}}}{1!}N \\ &= M + kN. \end{aligned}$$

Let $k=0$

$$I = M$$

let $k=1$

$$A = M + N.$$

Solving for N and N. From the first equation we get

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From the second equation we obtain

$$\begin{aligned} N &= A - M \\ &= \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}. \end{aligned}$$

Finally,

$$A^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

5. Computing e^{At} from A^k

Now that we have an explicit formula for A^k, we may now use it to create a compact formula of the matrix exponential. In lecture, we learned that

$$A^k = \sum_{r=1}^{R} \sum_{m=0}^{M_r-1} B_{r,m} \varphi_{m,a_r}(k),$$

which will be used to find e^{At}.

Lemma 5.1. *Let l be a natural number and a and t be real numbers. Show $\boldsymbol{D}^l(ye^{at}) = ((\boldsymbol{D}+a)^l y)e^{at}$.*

Proof.

$$\mathbf{D}^l(ye^{at}) = \sum_{k=0}^{l} \binom{l}{k} (\mathbf{D}^{l-k} y) \mathbf{D}^k e^{at}$$

Knowing that $\mathbf{D}^k e^{at} = a^k e^{at}$

$$\begin{aligned} \mathbf{D}^l(ye^{at}) &= \sum_{k=0}^{l} \binom{l}{k} (\mathbf{D}^{l-k} y)(a^k e^{at}) \\ &= ((\mathbf{D}+a)^l y)e^{at} \end{aligned}$$

□

Remark 5.2. By letting $y = t^j$ we can prove the next proposition.

Proposition 5.3.

$$\left[((\boldsymbol{D}+a)^l t^j) e^{at} \right] \Big|_{t=0} = ((\boldsymbol{D}+a)^l t^j) \Big|_{t=a}$$

Proof. Working with the Left Hand Side first we can derive the following:

$$\begin{aligned}\left[((\mathbf{D}+a)^l t^j)e^{at}\right]\Big|_{t=0} &= \sum_{k=0}^{l}\binom{l}{k}\mathbf{D}^k t^j a^{l-k}\Big|_{t=0} \\ &= \sum_{k=0}^{l}\binom{l}{k}a^{l-k}(\mathbf{D}^k t^j)\Big|_{t=0} \\ &= \begin{cases} 0 & \text{if} \neq \text{j or l< j} \\ \binom{l}{k}a^{l-j}j! & \text{if k = j} \\ \frac{a^{l-j}l!}{(l-j)!} & \text{if l = j} \end{cases}\end{aligned}$$

Working with the Right Hand Side we can derive the following:

$$\begin{aligned}((\mathbf{D}+a)^l t^j)\Big|_{t=0} &= \mathbf{D}^j t^l\Big|_{t=a} \\ &= \begin{cases} lt^{l-1} & \text{if j = 1} \\ l(l-1)t^{l-2} & \text{if j =2} \\ \vdots & \vdots \\ \frac{l!}{(l-j)!}a^{l-j} & \text{for any j} \end{cases}\end{aligned}$$

□

Proposition 5.4. *Using A^k, we define*

$$e^{At} = \sum_{r=1}^{R}\sum_{m=0}^{M_r-1} M_{r,m}(t^m e^{a_r t})$$

where R is the number of distinct eigenvalues, M_r is the multiplicity associated to the eigenvalue a_r, and $M_{r,m}$ is the associated coefficient matrix.

Proof. Knowing that

$$A^k = \sum_{r=1}^{R}\sum_{m=0}^{M_r-1} B_{r,m}\varphi_{m,a_r}(k),$$

we will then manipulate the equation by adding a summation, multiplying by t^k, and dividing by $k!$ to both sides.

$$\sum_{k=0}^{\infty}\frac{t^k A^k}{k!} = \sum_{r=1}^{R}\sum_{m=0}^{M_r-1} B_{r,m}\sum_{k=0}^{\infty}\frac{t^k\varphi_{m,a_r}(k)}{k!}$$

From calculus, we know we can rewrite this as:

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \sum_{r=1}^{R} \sum_{m=0}^{M_r-1} B_{r,m} \sum_{k=0}^{\infty} \frac{t^k \varphi_{m,a_r}(k)}{k!}$$

By Lemma 2.10, we can rewrite this as:

$$\begin{aligned} e^{At} &= \sum_{r=1}^{R} \sum_{m=0}^{M_r-1} B_{r,m} \sum_{k=0}^{\infty} \frac{\frac{t^k D^m t^k}{m!}}{k!}\Bigg|_{t=a_r} \\ &= \sum_{r=1}^{R} \sum_{m=0}^{M_r-1} \frac{B_{r,m}}{m!} \sum_{k=0}^{\infty} \frac{t^k D^m t^k}{k!}\Bigg|_{t=a_r} \end{aligned}$$

where$\dfrac{B_{r,m}}{m!} = M_{r,m}$

$$e^{At} = \sum_{r=1}^{R} \sum_{m=0}^{M_r-1} M_{r,m} \sum_{k=0}^{\infty} \frac{t^k D^m t^k}{k!}\Bigg|_{t=a_r}$$

By Proposition 5.3, we know

$$\begin{aligned} e^{At} &= \sum_{r=1}^{R} \sum_{m=0}^{M_r-1} M_{r,m} \sum_{k=0}^{\infty} \frac{t^k D^k (t^m e^{a_r t})}{k!}\Bigg|_{k=0} \\ &= \sum_{r=1}^{R} \sum_{m=0}^{M_r-1} M_{r,m} (t^m e^{a_r t}) \end{aligned}$$

In the following example we will solve for e^{At} by the Laplace Transform using Fulmer's Method, which we learned in class.

Example 5.5. Find e^{At} if

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

First, we will compute $sI - A$.

$$sI - A = \begin{bmatrix} s-2 & 1 \\ -1 & s \end{bmatrix}$$

Next, we will find the characteristic polynomial by computing the determinant and evaluating it at 0 to find the eigenvalues,s.

$C_A = det(sI - A) = (s-1)^2 = 0.$

therefore $s = 1, 1$

From class, we also know that the following is the basis for this particular matrix.

$$\mathcal{B}_{C_A} = \{e^t, te^t\}$$

Now, we can write an equation for e^{At}, where M and N are some matrices, as followed:

$$e^{At} = e^t M + te^t N$$

Next we need to find these particular matrices. So, we will set up of a system of equations by taking the derivative.

$$e^{At} = e^t M + te^t N$$
$$Ae^{At} = e^t M + (e^t + te^t)N$$

When $t = 0$, we get the following:

$$I = M + 0$$
$$A = M + N$$

So,

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } N = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Finally, we see that

$$e^{At} = e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^t \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Now, we will show a quicker way to find e^{At} by knowing A^k from the $\mathcal{Z}$-Transform.

Example 5.6. Find e^{At} if

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

From a previous example, we found that

$$A^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Next, we can manipulate this equation by adding a summation, multiplying by t^k, and dividing by $k!$ to both sides. We get the following:

$$\sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{k!} + \sum_{k=0}^{\infty} \frac{kt^k \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}}{k!}$$

From calculus, we know we can rewrite this as:

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{k!} + \sum_{k=0}^{\infty} \frac{kt^k \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}}{k!}$$

Since the matrices are not dependent on k we can move them out of the summation.

$$\begin{aligned} e^{At} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sum_{k=0}^{\infty} \frac{kt^k}{k!} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sum_{k=1}^{\infty} \frac{t^k}{(k-1)!} \end{aligned}$$

Again, from calculus we know we can rewrite the summations as a Taylor Series, giving us the following equation:

$$e^{At} = e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^t \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

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$\square$

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