Math Tune-Up
Louisiana State University
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Lectures on Partial Differential Equations and Hilbert Space

1. A linear partial differential equation of physics

We begin by considering the simplest mathematical model of conduction of electricity in a material body. This is a part of *electrostatics*. In the body, there is an electric field $\mathbf{E}$ at equilibrium, which gives rise to a static electric current $\mathbf{J}$ (steady movement of charge). The electric field is conservative, meaning that its line integral about any closed curve in the body vanishes. Equivalently, it is generated by a scalar *potential* u:

$$\mathbf{E} = -\nabla u. \tag{1.1}$$

The electric field exerts a force on charged particles in the medium, causing a current density $\mathbf{J}$. The current and the electric field are related through a (real-valued) tensor σ; this is an example of a *constitutive relation*:

$$\mathbf{J} = \sigma \mathbf{E}. \tag{1.2}$$

In a static situation, the charge density ρ (charge per volume) is constant in time. This means that the charge flowing out of any region R of the material through its boundary ∂R must balance the charge entering into the body through an external source, which we denote by f (charge density per time):

$$\int_{\partial R} \mathbf{J} \cdot n \, dS = \int_R f. \tag{1.3}$$

Since this holds for each region R, the Divergence Theorem $\int_{\partial R} \mathbf{J} \cdot n \, dS = \int_R \nabla\cdot \mathbf{J} \, dV$, gives

$$-\nabla\cdot \mathbf{J} = f. \tag{1.4}$$

In summary, the fields u, $\mathbf{E}$, $\mathbf{J}$, and f are related as follows:

$$u = \text{electric potential} \quad \text{(energy/charge)}, \tag{1.5}$$

$$\mathbf{E} = -\nabla u = \text{electric field} \quad \text{(force/charge)}, \tag{1.6}$$

$$\mathbf{J} = \sigma \mathbf{E} = -\sigma \nabla u = \text{current density} \quad \text{(charge/(time·area))}, \tag{1.7}$$

$$f = -\nabla\cdot \sigma \nabla u = \text{source} \quad \text{(charge/(time·volume))}. \tag{1.8}$$

The final equation is our inhomogeneous, linear, scalar, second-order, divergence-form, elliptic, partial differential equation:

$$-\nabla\cdot \sigma \nabla u = f. \tag{1.9}$$

This equation can also be obtained through a *variational principle*: it is the "Euler-Lagrange" equation corresponding to a certain "Lagrangian density". We will see that this method provides valuable insight into the understanding and solution of a PDE. We shall use some examples to elucidate this principle.

1.1 The Dirichlet problem

This problem is to find the solution to the conductivity equation (1.9) in a bounded domain Ω in $\mathbb{R}^3$ subject to the condition that the potential u be fixed at values given by a function g defined on the boundary of Ω:

$$-\nabla\cdot\sigma\nabla u = f \qquad \text{in } \Omega, \tag{1.10}$$

$$u = g \qquad \text{on } \partial\Omega. \tag{1.11}$$

To derive this system, let us begin with the Lagrangian density associated with an arbitrary electric field in the material occupying the region Ω. If u is the (real-valued) potential, then we denote this density, as a function of x, by $L[u]$, and define it by

$$L[u](x) = \tfrac{1}{2}\sigma|\nabla u|^2 - fu \quad \text{(power/volume)}. \tag{1.12}$$

The first term is equal to half the product $\mathbf{E}\cdot\mathbf{J}$ of the electric field and the current. It has units of power density. Using L, we define the Lagrangian functional $\mathcal{L}(u)$ for functions u defined on Ω:

$$\mathcal{L}[u] = \int_\Omega L[u](x)\,dV(x) = \int_\Omega \left[\tfrac{1}{2}\sigma|\nabla u|^2 - fu\right] dV. \tag{1.13}$$

We now consider the variation of $\mathcal{L}[u]$ as it depends on the function u. As we are considering functions u that satisfy $u|_{\partial\Omega} = g$, we allow u to vary in the directions of functions v such that $v|_{\partial\Omega} = 0$. The symbol $\delta\mathcal{L}/\delta u|_{u_0}(v)$ denotes the variational derivative of $\mathcal{L}[u]$ with respect to variations of u at the function u_0 in the direction of v. We compute that, for sufficiently smooth functions,

$$\begin{aligned}\left.\frac{\delta\mathcal{L}}{\delta u}\right|_u (v) := \lim_{h\to 0}\frac{1}{h}\left[\mathcal{L}(u+hv) - \mathcal{L}(u)\right] &= \int_\Omega \left[\sigma\nabla u\cdot\nabla v - fv\right] dV \\ &= \int_\Omega \left[-\nabla\cdot\sigma\nabla u - f\right] v\,dV. \end{aligned} \tag{1.14}$$

Now let us seek a critical function u of $\mathcal{L}$, that is, set $\delta\mathcal{L}/\delta u$ equal to zero. This means that we seek a function u such that, for all v (equal to zero on $\partial\Omega$), (1.14) vanishes, in other words,

$$\nabla\cdot\sigma\nabla u + f = 0, \tag{1.15}$$

which is exactly equation (1.10). The boundary condition $u|_{\partial\Omega} = g$ is enforced separately.

As it is written, it is required that $\sigma\nabla u$ be differentiable. But the law expressed by (1.10) is a merely a more restrictive form of the law of conservation of charge expressed by (1.3), namely

$$\int_{\partial R} \sigma\nabla u\cdot n\,dS = \int_R f, \tag{1.16}$$

for all regions R whose closure is contained in Ω. Notice that only one derivative of u is required for this formulation. Using similar reasoning, we might as well go back to the expression (1.14) for the variation of $\mathcal{L}$ *before* integration by parts, and formulate what is called the "weak form" of the differential equation (1.15):

$$\int_\Omega \left[\sigma\nabla u\cdot\nabla v - fv\right] dV = 0 \quad \text{for all } v\in C_0^\infty(\bar{\Omega}). \tag{1.17}$$

$$u = g \quad \text{on } \partial\Omega. \tag{1.18}$$

This form turns out to be much more useful, for it does not require that the current $\sigma\nabla u$ be differentiable. It also allows the regularity of the tensor σ and the source f to be relaxed significantly.

Now, what about the existence of a solution to this PDE? Because of the convexity of the functional $L[u]$ and the convexity of the set of admissible functions, that is, those that satisfy the boundary condition, if $L[u]$ admits a critical function, this function must minimize $L[u]$. However, the set of functions that are infinitely differentiable is typically not big enough to include such a function; we have to complete this space somehow. It turns out that the set of functions that are natural candidates for solutions to the *weak* form of the PDE (the weakly differentiable functions as described in Section 2) is large enough to provide a unique solution, even for very irregular conductivity tensors and sources.

1.2 The Neumann problem

This is the conductivity problem in which, instead of fixing the values of the potential on the boundary, the current directed out of the boundary, or $-\sigma\nabla u \cdot n$, is fixed:

$$-\nabla\cdot\sigma\nabla u = f \qquad \text{in } \Omega, \tag{1.19}$$

$$-\sigma\nabla u \cdot n = g \qquad \text{on } \partial\Omega. \tag{1.20}$$

The Lagrangian density for this system is defined for functions that have well-defined boundary values. It has two parts. One part is supported in the interior of Ω and coincides with that of the Dirichlet problem, and the other part is supported on the boundary of Ω:

$$L[u] = \tfrac{1}{2}\sigma|\nabla u|^2 - fu + gu\,\delta_{\partial\Omega} \quad \text{(power/volume)}. \tag{1.21}$$

The Lagrangian functional is the integral of $L[u]$ over the closure of Ω:

$$\mathcal{L}[u] = \int_\Omega \left[\tfrac{1}{2}\sigma|\nabla u|^2 - fu\right] dV + \int_{\partial\Omega} gu\, dS. \tag{1.22}$$

In the Neumann case, we allow the function u to vary over functions on Ω with arbitrary boundary values. Setting the variational derivative of $\mathcal{L}$ equal to zero for all variations (with no restrictions on the boundary), gives the system (1.19,1.20). The part of the Lagrangian arising from the boundary gives rise to the boundary condition on the outward flux. The weak formulation of the Neumann problem is

$$\int_\Omega [\sigma\nabla u \cdot \nabla v - fv]\, dV + \int_{\partial\Omega} gv\, dS \;=\; 0 \quad \text{for all } v \in C^\infty(\bar{\Omega}). \tag{1.23}$$

2. Hilbert space

2.1 Infinite-dimensional linear spaces

Let Ω be an (open) domain in $\mathbb{R}^d$, and let us consider the linear space of complex-valued functions defined on the closure $\bar{\Omega}$ of Ω that are infinitely many times differentiable. This space is denoted by $C^\infty(\bar{\Omega})$ (or C^∞ for short), and it is certainly a vector space, or linear space. Moreover, it does not possess a finite basis, and is therefore infinite-dimensional.

A basis for an infinite-dimensional linear space is sometimes called a Hamel basis. This is a linearly independent set of elements of the vector space such that each element of the space is equal to a unique linear combination of a finite subset of elements from the basis. It is purely an *algebraic* object, and, as such, infinite combinations have no meaning in this context. Infinite combinations (series) require the added analytic structure of a limit, obtained by endowing the space with a metric.

2.2 Metric, norm, and inner product

2.2.1 *The mean-square norm and its abstraction*

As we have seen in the Lagrangian functionals above, the integral of the square of a function plays an important role. Let us denote by V the linear subspace of C^∞ consisting of those functions f such that

$$\int_\Omega |f|^2 < \infty. \tag{2.24}$$

This allows us to define a *norm* (the mean-square norm) on V by means of

$$\|f\| = \left(\int_\Omega |f|^2 \right)^{1/2}. \tag{2.25}$$

The real-valued function $\|\cdot\|$ possesses the defining properties of an (abstract) norm:

$$\|f\| \geq 0, \text{ with equality if and only if } f = 0, \tag{2.26}$$

$$\|cf\| = |c|\|f\|, \tag{2.27}$$

$$\|f + g\| \leq \|f\| + \|g\| \quad \text{(triangle inequality)}. \tag{2.28}$$

A linear space endowed with a norm is called a *normed linear space*. A norm defined on a linear space induces a more general structure called a *metric*. A metric provides a way to express a "distance" between elements of the space. A metric d is induced by a norm $\|\cdot\|$ by

$$d(f, g) := \|f - g\|. \tag{2.29}$$

An (abstract) metric possesses, by definition, the following properties:

$$d(f, g) \geq 0, \text{ with equality if and only if } f = g, \tag{2.30}$$

$$d(f, g) = d(g, f), \tag{2.31}$$

$$d(f, h) \leq d(f, g) + d(g, h) \quad \text{(triangle inequality)}. \tag{2.32}$$

A metric does not require an algebraic structure on the space in which it is defined. A set endowed with a metric is called a *metric space*. A subset of $\mathbb{R}^d$, for example, is a metric space, in which the natural metric is a restriction of the usual one on $\mathbb{R}^d$ to the subset. A metric space is thus a more general structure than a normed linear space.

The weak formulation (1.17) of the Dirichlet problem and the last expression in (1.14) suggests the introduction of the structure of an *inner product*, which is a pairing $\langle \cdot, \cdot \rangle$ of functions from V, producing a complex number. A natural inner product on the space V is

$$\langle f, g \rangle := \int_\Omega f\bar{g}\,, \tag{2.33}$$

in which the bar denotes complex conjugation. We must first make sure that this pairing is well defined on V. Indeed, if $f, g \in V$, then the pointwise application of the rule $2ab < a^2+b^2$ yields

$$2\int |fg| \le \int |f|^2 + \int |g|^2 < \infty. \tag{2.34}$$

This pairing can be called a (complex) inner product (in a linear space over $\mathbb{C}$) because it satisfies the following properties required of an (abstract) inner product:

$$\langle f_1+f_2, g\rangle = \langle f_1, g\rangle + \langle f_2, g\rangle, \tag{2.35}$$
$$\langle f, g_1+g_2\rangle = \langle f, g_1\rangle + \langle f, g_2\rangle, \tag{2.36}$$
$$\langle cf, g\rangle = c\langle f, g\rangle, \tag{2.37}$$
$$\langle f, cg\rangle = \bar{c}\langle f, g\rangle, \tag{2.38}$$
$$\langle g, f\rangle = \overline{\langle f, g\rangle}, \tag{2.39}$$
$$\langle f, f\rangle > 0 \quad \text{if } f \neq 0. \tag{2.40}$$

These properties are not independent; certain of them can be derived from others (for example, one can remove the second and fourth properties and derive them from the first, third, and fifth). A linear space endowed with an inner product is called an *inner product space.*

The inner product (2.33) on V that we defined above gives rise to the norm (2.25) in a simple way,

$$\|f\|^2 = \langle f, f\rangle. \tag{2.41}$$

In fact, by this rule, each inner product space is also a normed linear space. We have seen, then, that an inner product is a stronger structure than a norm and a norm is a stronger structure than a metric. The first two of these subsume a vector space structure.

There arises an interesting question: What property must a norm possess in order that it be induced by an inner product? The answer is given by the "parallelogram law", which is easy to prove for a norm that arises from an inner product.

$$2\|f\|^2 + 2\|g\|^2 = \|f+g\|^2 + \|f-g\|^2 \quad \text{(parallelogram law)}. \tag{2.42}$$

However, this law is not satisfied by all norms. But a *complex* normed linear space in which it is satisfied can be endowed with a unique inner product which gives rise to the norm. That inner product is obtained from the norm by the so-called "polarization identity":

$$\langle f, g\rangle = \tfrac{1}{4}\left[\|f+g\|^2 - \|f-g\|^2 + i\|f+ig\|^2 - i\|f-ig\|^2\right]. \tag{2.43}$$

2.2.2 *The mean-square-gradient norm*

Another norm on $C^\infty(\bar{\Omega})$ that is perhaps more manifestly related to the weak PDE formulations is one that measures the derivatives of functions. Let us denote by V_0' the space of functions f in $C^\infty(\bar{\Omega})$ *such that* $f|_{\partial\Omega} = 0$ *and endowed with the norm*

$$\|f\|_0' := \int_\Omega |\nabla f|^2 , \quad f \in V_0' . \tag{2.44}$$

This norm is induced by the inner product

$$\langle f, g\rangle_0' = \int_\Omega \nabla f \cdot \nabla \bar{g}\,, \quad f,\, g \in V_0'\,. \tag{2.45}$$

The formula (2.46) no longer defines a norm if we include *all* functions in C^∞. This is because all constant functions on Ω return a value of zero, and the first property (2.26) is violated (and we have only a *semi-norm*). This indicates that the value of the function must also contribute to any norm involving the gradient. Let us denote by V' the space of functions f in $C^\infty(\bar{\Omega})$ *endowed with the norm*

$$\|f\|' := \left(\int_\Omega \left(|f|^2 + |\nabla f|^2\right) \right)^{1/2}, \quad f \in V'\,. \tag{2.46}$$

This norm is induced by the inner product

$$\langle f, g\rangle' = \int_\Omega \left(f\bar{g} + \nabla f \cdot \nabla \bar{g}\right)\,, \quad f,\, g \in V'\,. \tag{2.47}$$

2.3 Completeness

Let us consider the following motivational scenario. We have a sequence of materials occupying the region Ω, with conductivities given by smooth tensor functions $\{\sigma_n\}$. The conductivities typically vary in space (perhaps there is a periodic structure at the microscopic level), but their values have a common lower bound and a common upper bound. Everything else in the experiment being the same (boundary values, sources) we obtain solutions u_n satisfying the conductivity equation, and they obey the common bound in the mean-square-gradient norm,

$$\|u_n\|' < M \quad \text{for all } n. \tag{2.48}$$

There is a theorem that states that such a sequence admits a subsequence $\{u_{n_j}\}_{j=1}^\infty$ that is Cauchy in the mean-square norm,

$$\|u_{n_j} - u_{n_i}\| \to 0 \quad \text{as } i, j \to \infty. \tag{2.49}$$

Now the question is, does $\{u_{n_j}\}$ admit a limit in V? That is, is there a function $u \in V$ such that $\|u - u_{n_j}\| \to 0$ as $n \to \infty$? Not in general. In order to find "the limit" of the subsequence, we must relax the conditions on the admissible solution functions.

Another equally important reason to relax the conditions on our functions is that the conductivities σ_n may only be piecewise continuous, and the solution to a conductivity problem will not even have continuous derivatives. In the sequel, we show how to complete the spaces V, V', etc., in order to obtain an adequate theory for solving practical problems in conductivity. The outcome proves to be have much deeper implications than meet the eye in the theory of unbounded operators and elliptic partial differential equations.

The abstract way to obtain a limit of a Cauchy sequence in a normed linear space, say V, is to form the completion of the space. We will not go into the details of this construction, but the idea is that each Cauchy sequence in V that does not already have a limit gives rise to a new element, which will serve as its limit point. Two Cauchy sequences $\{u_i\}$ and $\{v_i\}$

such that $(u_i - v_i) \to 0$ as $i \to \infty$ give rise to the same element. The union of V and all the new elements form the *completion* $\bar{V}$ of V. The norm of V is extended in such a way that the closure of V is a normed linear space in which each Cauchy sequence has a limit point. Such a space is called *complete*. (More generally, one can form the completion of a metric space.) If V is an inner product space, then $\bar{V}$ is also an inner product space, in which the inner product is the natural extension of that on V. A complete normed linear space is called a *Banach space*, and a complete inner product space is called a ***Hilbert space***.

So a (complex) Hilbert space is simply a vector space (over $\mathbb{C}$*) that is endowed with an inner product and that is complete with respect to the norm induced by the inner product.* Each finite-dimensional inner product space is complete and is therefore a Hilbert space. Infinite-dimensional inner product spaces are not necessarily complete; in fact, as we have seen, this point is the motivation for this section.

2.3.1 *Completion of* C^∞ *in the mean-square norm*

Now here is the important consideration. Although a Cauchy sequence in V ($C^\infty(\bar{\Omega})$ with the mean-square norm) does have a limit in the abstract completion $\bar{V}$, this limit will only be meaningful if it can be realized as a genuine function on Ω that is square-integrable. Is such a realization possible? The answer is affirmative. In order to construct these limit functions and thereby realize $\bar{V}$ as a true function space, we must discover a more general notion of integration. The appropriate notion in our context is that of *Lebesgue integration*. Its full development requires quite a bit of analysis, but it is based on constructing functions as *(upper) limits of increasing sequences of step functions*.

$$s_n(x) \nearrow f(x) \text{ as } n \to \infty \quad \text{for almost all } x \text{ and each } s_n \text{ is a step function,} \tag{2.50}$$

$$\int s_n \nearrow \int f \text{ as } n \to \infty \quad (\text{definition of } \textstyle\int f). \tag{2.51}$$

Since each step function (on a bounded domain) has a well-defined integral and the value of $\lim_{n\to\infty} \int s_n$ is independent of the sequence s_n used to construct f (this must be proved), the integral of f, as given by (2.51) is well defined. One also constructs *(lower) limits of decreasing sequences of step functions* and their integrals. All of the functions constructed in this manner and their sums form the linear space of *Lebesgue-integrable functions*. The strange terminology "for almost all x" in (2.50) is a necessary technicality in the theory of Lebesgue integration. Something that is true almost everywhere is true for all x except for those x in a set that can be covered by a sequence of open balls whose total length is less than any arbitrarily prescribed number (no matter how small). Such a set is called a *set of measure zero*. The Lebesgue integral is extended to unbounded domains Ω in a straightforward way.

The space of all complex-valued functions on Ω that are square-integrable in the Lebesgue sense is a complex linear space, and it is endowed with the inner product and norm given by the same formulas (2.25,2.33) as those for V. The resulting inner-product space is denoted by $L^2(\Omega)$, and it contains V as a sub-inner-product space. Two facts about $L^2(\Omega)$ are important for us:

- $L^2(\Omega)$ is complete. It is therefore a Hilbert space.
- $C^\infty(\bar{\Omega})$ is dense in $L^2(\Omega)$. This means that each function $f \in L^2(\Omega)$ admits a sequence $\{f_n\}$ from $C^\infty(\bar{\Omega})$ that converges to f in the L^2-sense, that is, $\|f - f_n\| \to 0$ as $n \to \infty$.

The first statement fulfills our desire that each Cauchy sequence in V admit a genuine limit function, although we do not expect this function to be smooth, or even continuous, in general. The second statement, in light of the first, tells us that $L^2(\Omega)$ represents concretely the completion of V (and no more and no less).

Because of the technicality of defining Lebesgue-integral functions, in particular, the issue of convergence of step functions for almost-all x, or *convergence almost everywhere*, one has to make certain that the mean-square norm really does satisfy the properties of a norm (p. 4). It turns out that it is technically not a norm, but if we identify functions that are *equal almost everywhere*, then it is. This means that, if f and g are Lebesgue-square-integrable and $f(x) = g(x)$ for all x except those in a set of measure zero, then they are considered to represent the same element of $L^2(\Omega)$.

2.3.2 *Completion of C^∞ in the mean-square-gradient norm*

The completion of C^∞ in the mean-square norm is L^2. Now what about its completion in the mean-square-gradient norm? It is clear that a not only a generalization of integration, but also a generalization of differentiation (forming the gradient of a function) will have to be made. We can form the abstract completion of V', but again, the limiting elements obtained from Cauchy sequences in V' need to be genuine functions with gradients. Suppose we have a Cauchy sequence $\{f_n\}$ in V', that is,

$$\|f_n - f_m\|'^2 = \int_\Omega |f_n - f_m|^2 + \int_\Omega |\nabla f_n - \nabla f_m|^2 \to 0 \quad \text{as } m, n \to \infty. \tag{2.52}$$

It follows that $\|f_n - f_m\| \to 0$ and $\|\nabla f_n - \nabla f_m\| \to 0$ also. What we have learned already is that there is a scalar function f in L^2 and a vector function F in $(L^2)^d$ such that

$$\|f - f_n\| \to 0 \text{ as } n \to \infty, \tag{2.53}$$

$$\|F - \nabla f_n\| \to 0 \text{ as } n \to \infty. \tag{2.54}$$

The question is, can F be considered to be the gradient of f in a relaxed sense that subsumes the notion of the classical gradient? The answer is affirmative. To see how to relax the definition of the gradient, we observe that, for each smooth vector field Φ with compact support contained in Ω, we have

$$\int_\Omega \Phi \cdot \nabla f = -\int_\Omega (\nabla\cdot\Phi) f\,. \tag{2.55}$$

Now, one can prove that the integrals $\int_\Omega \Phi \cdot \nabla f$, for all Φ, completely determine ∇f. Therefore, even if f is a function for which ∇f is not defined classically, we may still be able to use $-\int_\Omega (\nabla\cdot\Phi) f$, which is always well defined, to define a "weak gradient". *We say that f* **is weakly differentiable in** L^2 *if there exists a vector field F in $\left(L^2(\Omega)\right)^2$ such that, for each smooth vector field Φ with compact support contained in Ω, we have*

$$\int_\Omega \Phi \cdot G = -\int_\Omega (\nabla\cdot\Phi) f\,. \tag{2.56}$$

Notice that G may not be the classical gradient of any function. In fact, if it is, then it will be the gradient of f. So we have broadened our set of functions that have gradients. The space of all L^2 functions that possess a weak L^2 gradient is a linear space, and it is endowed with the norm and inner product given by (2.46,2.45). The resulting inner-product space is denoted by $W^{1,2}(\Omega)$ (one derivative in L^2) or H^1. Again, two important facts can be proved:

- $W^{1,2}(\Omega)$ is complete. It is therefore a Hilbert space.
- $C^\infty(\bar{\Omega})$ is dense in $W^{1,2}(\Omega)$. This means that each function $f \in W^{1,2}(\Omega)$ admits a sequence $\{f_n\}$ from $C^\infty(\bar{\Omega})$ that converges to f in the norm-square-gradient sense, that is, $\|f - f_n\|'^2 \to 0$, or, equivalently, $\|f - f_n\|$ and $\|\nabla f - \nabla f_n\|$ both tend to zero as $n \to \infty$. Here, ∇ refers to the operation of taking the weak derivative.

Because of the first statement, we are able to identify with any Cauchy sequence in V' a limit function, whose gradient is understood in the weak sense. In addition, because of the second statement, the space $W^{1,2}(\Omega)$ is nothing less and nothing more than a realization of the completion of V': it consists of exactly all limit elements of Cauchy sequences in V'. The proofs of these statements require some good math.

3. Existence and uniqueness

At this point, we are ready to discuss the existence and uniqueness of solutions to the weak-form PDEs (1.17) and (1.23). However, these notes are already too long for two lectures. The idea is that one should look for solutions in the space $W^{1,2}$ because the weak forms are expressed in terms of (generalized) inner products in that space that are bounded from below and above if σ is likewise bounded on all of Ω. The pertinent theorem is the *Lax-Milgram Theorem*, which is a theorem on bounded complex-bilinear forms in Hilbert space like those appearing in the weak forms we have seen. By placing the PDE in the context of this theorem, one obtains results on existence and uniqueness.

4. Exercises

1. Derive the formula (1.14).

2. Derive the system (1.19,1.20) by seeking a critical function u of the Lagrangian functional $\mathcal{L}[u]$ (1.22).

3. Prove that $C^\infty(\bar{\Omega})$ is an infinite-dimensional linear space.

4. Prove that d defined below is a metric on the set S of binary sequences,

$$S = \big\{ s = \{s_0, s_1, s_2, \dots\} : s_i \in \{0, 1\} \text{ for } i = 1, \dots, \infty \big\}, \tag{4.57}$$

$$d(s,t) = \sum_{i=1}^{\infty} \frac{|s_i - t_i|}{2^i}. \tag{4.58}$$

5. Prove the "polarization identity" (2.43).

6. Prove that (2.44) and (2.46) define norms.

7. Prove that $C^\infty(\bar{\Omega})$ is not complete in the mean-square norm.

8. Prove that each finite-dimensional normed linear space is complete.

9. Prove that the constant function 1 on the interval $[0, 1]$ and the function on $[0, 1]$ that is equal to 1 at the irrational numbers and 0 at the rational numbers both represent the same element of $L^2([0, 1])$.

10. Derive equation (2.53).