Calculating A^k using Fulmer's Method

Rasheen Alexander, Katie Huston, Thomas Le, Camera Whicker

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Why find A^k and e^{At}?

- A^k is essential to find the solutions to difference equations.
- Calculating e^{At}, the matrix exponential.
- e^{At} is used in solving matrix linear differential equations.

Definition

Let n be a nonnegative integer. The **falling factorial** is the sequence $k^{\underline{n}}$, with $k = 0, 1, 2, \ldots$ given by the following formula.

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Let n be a nonnegative integer. The **falling factorial** is the sequence $k^{\underline{n}}$, with $k = 0, 1, 2, \ldots$ given by the following formula.

$$k^{\underline{n}} = k(k-1)(k-2)\cdots(k-n+1).$$

If k were allowed to be a real variable then $k^{\underline{n}}$ could be characterized as the unique monic polynomial of degree n that vanishes at $0, 1, \ldots, n-1$. Observe also that $k^{\underline{n}}|_{k=n} = n!$.

Definition

Let n be a nonnegative integer and a be a complex number. We define a sequence $\varphi_{n,a}(k)$ as

$$\varphi_{n,a}(k) = \begin{cases} \frac{a^{k-n} k^{\underline{n}}}{n!} & a \neq 0, \\ \delta_n(k) & a = 0. \end{cases}$$

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where

$$\delta_n(k) = \begin{cases} 0 & k \neq n \\ 1 & k = n \end{cases}$$

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With $\varphi_{0,0}(k), \varphi_{1,4}(k)$, and $\varphi_{2,5}(k)$, we have

$$\begin{aligned} \varphi_{0,0}(k) &= \delta_0(k) &&= (1, 0, 0, 0, 0, \ldots) \\ \varphi_{1,4}(k) &= 4^{k-1}k &&= (0, 1, 8, 48, 256, \ldots) \end{aligned}$$

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\varphi_{1,4}(k) &= 4^{k-1}k &= (0,1,8,48,256,\ldots) \\
\varphi_{2,5}(k) &= \frac{5^{k-2}k(k-1)}{2} &= (0,0,1,15,150,\ldots)
\end{aligned}$$

Definition

Let $y(k)$ be a sequence of complex numbers. We define the $\mathcal{Z}$-**transform** of $y(k)$ to be the function $\mathcal{Z}\{y(k)\}(z)$, where z is a complex variable, by the following formula:

$$\mathcal{Z}\{y(k)\}(z) = \sum_{k=0}^{\infty} \frac{y(k)}{z^k}$$

Now that we've defined the $\mathcal{Z}$-Transform, we can now apply it to $\varphi_{n,a}$. Let $a \in \mathbb{C}$ and $n \in \mathbb{N}$ and we obtain the following formula.

$$\mathcal{Z}\{\varphi_{n,a}(k)\}(z) = \frac{z}{(z-a)^{n+1}}$$

Theorem

Let A be an n x n matrix with entries in the complex plane. Then

$$\mathcal{Z}\{A^k\}(z) = z(zI - A)^{-1}$$

where I is the n x n identity matrix.

Note that the $\mathcal{Z}$-Transform is one-to-one and linear. Therefore, the $\mathcal{Z}$-Transform has an inverse.

Now that we know that the $\mathcal{Z}$-Transform is invertable we obtain the following formula

$$A^k = \mathcal{Z}^{-1}\{z(zI - A)^{-1}\}$$

Example

Find A^k if

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

Recall $\mathcal{Z}\{A^k\}(z) = z(zI - A)^{-1}$.
First, we would compute $zI - A$.

$$zI - A = \begin{bmatrix} z-2 & 1 \\ -1 & z \end{bmatrix}$$

Next, compute the inverse of $zI - A$.

$$\begin{aligned}(zI - A)^{-1} &= \frac{1}{z^2 - 2z + 1}\begin{bmatrix} z & -1 \\ 1 & z-2 \end{bmatrix} \\ &= \frac{1}{(z-1)^2}\begin{bmatrix} z & -1 \\ 1 & z-2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{z}{(z-1)^2} & \frac{-1}{(z-1)^2} \\ \frac{1}{(z-1)^2} & \frac{z-2}{(z-1)^2} \end{bmatrix}\end{aligned}$$

So we must perform partial fraction decomposition to obtain

$$\frac{z}{(z-1)^2} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2}$$

$$z = A(z-1) + B$$

If $z = 1$,

$$1 = A(1-1) + B$$
$$1 = A(0) + B$$
$$1 = 0 + B$$
$$B = 1$$

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If $z = 0$,

$$0 = A(0-1) + B$$
$$0 = A(-1) + B$$
$$0 = -A + B$$
$$A = B$$
$$A = 1$$

Plug in results:

$$\frac{z}{(z-1)^2} = \frac{1}{(z-1)} + \frac{1}{(z-1)^2}$$

$$\frac{z-2}{(z-1)^2} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2}$$

$$z-2 = A(z-1) + B$$

If $z = 1$,

$$1-2 = A(1-1) + B$$
$$-1 = A(0) + B$$
$$-1 = 0 + B$$
$$B = -1$$

$$\frac{z-2}{(z-1)^2} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2}$$

$$z-2 = A(z-1) + B$$

If $z = 0$,

$$\begin{aligned} 0-2 &= A(0-1) + B \\ -2 &= A(-1) + B \\ -2 &= -A + B \\ 2 + B &= A \\ 2 + (-1) &= A \\ 1 &= A \end{aligned}$$

Plug in results:

$$\frac{z-2}{(z-1)^2} = \frac{1}{(z-1)} + \frac{-1}{(z-1)^2}$$

$$(zI - A)^{-1} = \begin{bmatrix} \frac{1}{(z-1)} + \frac{1}{(z-1)^2} & \frac{-1}{(z-1)^2} \\ \frac{1}{(z-1)^2} & \frac{1}{(z-1)} + \frac{-1}{(z-1)^2} \end{bmatrix}$$

Finally, we obtain the following:

$$(zI - A)^{-1} = \frac{1}{z-1}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \frac{1}{(z-1)^2}\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}$$

which separates the original matrix into two matrices that have a common denominator for each entry.

Multiplying z into the equation, we obtain

$$z(zI - A)^{-1} = \frac{z}{z-1}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \frac{z}{(z-1)^2}\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}$$

By previous definition and taking the inverse $\mathcal{Z}$-transform, we find that

$$\varphi_{n,a}(k) = \mathcal{Z}^{-1}\left\{\frac{z}{(z-a)^{n+1}}\right\}$$

and since we have fractions that are of this form, we may apply the inverse Z-transform to obtain

$$\begin{aligned} A^k &= \mathcal{Z}^{-1}\left\{\frac{z}{z-1}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{z}{(z-1)^2}\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}\right\} \\ &= \varphi_{0,1}(k)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \varphi_{1,1}(k)\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix} \\ &= \frac{1^{k-0}k^{\underline{0}}}{0!}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{1^{k-1}k^{\underline{1}}}{1!}\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix} \\ &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + k\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix} \end{aligned}$$

Therefore, we obtain that

$$A^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Example

Find A^k by using the Fulmer's Method if

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

Recall $A^k = \mathcal{Z}^{-1}\{z(zI - A)^{-1}\}$. We first want to compute the determinant of $z(zI - A)^{-1}$. We start by computing $zI - A$.

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Lastly, we must multiply both sides by z

$$z\det((zI - A)^{-1}) = \frac{z}{(z-1)^2}$$

We can write A^k in the following way

$$A^k = \varphi_{0,1}(k)M + \varphi_{1,1}(k)N.$$

where M and N are our unknown matrices. By the way we defined the φ function, we obtain

$$\begin{aligned} A^k &= \frac{1^{k-0}k^{\underline{0}}}{0!}M + \frac{1^{k-1}k^{\underline{1}}}{1!}N \\ &= M + kN \end{aligned}$$

Let $k = 0$

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Solve for M and N. From the first equation, we get

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

From the second equation, we obtain

$$\begin{aligned} N &= A - M \\ &= \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

Finally,

$$A^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Theorem

The standard basis $\mathbb{B}_q = \{\varphi_{1\lambda_1}, \varphi_{1\lambda_2}, \cdots, \varphi_{2\lambda_1}, \varphi_{2\lambda_2}, \cdots \varphi_{r\lambda_m}\}$ *is linearly independent.*

Write $\mathbb{B}_q$ as a linear combination.

$$a_{11}\varphi_{1\lambda_1} + a_{12}\varphi_{1\lambda_2} + \cdots + a_{1m}\varphi_{1\lambda_m} +$$
$$a_{21}\varphi_{2\lambda_1} + a_{22}\varphi_{2\lambda_2} + \cdots + a_{2m}\varphi_{2\lambda_m} +$$
$$\vdots$$
$$a_{r1}\varphi_{r\lambda_1} + a_{r2}\varphi_{r\lambda_2} + \cdots + a_{rm}\varphi_{r\lambda_m} = 0$$

$$a_{11}\frac{z}{(z-\lambda_1)^2} + a_{12}\frac{z}{(z-\lambda_2)^2} + \cdots + a_{1m}\frac{z}{(z-\lambda_m)^2} +$$
$$a_{21}\frac{z}{(z-\lambda_1)^3} + a_{22}\frac{z}{(z-\lambda_2)^3} + \cdots + a_{2m}\frac{z}{(z-\lambda_m)^3} +$$
$$\vdots$$
$$a_{r1}\frac{z}{(z-\lambda_1)^{r+1}} + a_{r2}\frac{z}{(z-\lambda_2)^{r+1}} + \cdots + a_{rm}\frac{z}{(z-\lambda_m)^{r+1}} = 0$$

Regroup to form like denominators

$$\frac{a_{11}(z(z-\lambda_1)^{r-1}) + a_{21}(z(z-\lambda_1)^{r-2}) + \cdots + a_{r1}(z)}{(z-\lambda_1)^{r+1}} +$$

$$\frac{a_{12}(z(z-\lambda_2)^{r-1}) + a_{22}(z(z-\lambda_2)^{r-2}) + \cdots + a_{r2}(z)}{(z-\lambda_2)^{r+1}} +$$

$$\vdots$$

$$\frac{a_{1m}(z(z-\lambda_m)^{r-1}) + a_{2m}(z(z-\lambda_m)^{r-2}) + \cdots + a_{rm}(z)}{(z-\lambda_m)^{r+1}} = 0$$

Let $n_b(z - \lambda_b)$ be a polynomial.

$$\frac{n_1(z-\lambda_1)}{(z-\lambda_1)^{r+1}} + \frac{n_2(z-\lambda_2)}{(z-\lambda_2)^{r+1}} + \cdots + \frac{n_m(z-\lambda_m)}{(z-\lambda_m)^{r+1}} = 0$$

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$$\frac{n_1(z-\lambda_1)}{(z-\lambda_1)^{r+1}} + \frac{n_2(z-\lambda_2)}{(z-\lambda_2)^{r+1}} + \cdots + \frac{n_m(z-\lambda_m)}{(z-\lambda_m)^{r+1}} = 0$$

If $n_1(z-\lambda_1) \neq 0$, then

$$\lim_{z\to\lambda_1}\left[\frac{n_1(z-\lambda_1)}{(z-\lambda_1)^{r+1}} + \frac{n_2(z-\lambda_2)}{(z-\lambda_2)^{r+1}} + \cdots + \frac{n_m(z-\lambda_m)}{(z-\lambda_m)^{r+1}}\right] = \infty + C = 0$$

where C is a constant. Thus, we get a contradiction.

Therefore, $n_1(z - \lambda_1) = 0$ which implies that $n_1(z) = 0$ and $a_{11}, a_{12}, \cdots, a_{1m} = 0$. You can continue this argument by induction to obtain $\forall a's = 0$

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With this Theorem we may now show that we may apply Fulmer's method to any $n \times n$ matrix.

Definition

Let n be a natural number and $a(k)$ to be a sequence with complex terms. We define E as a **shift operator** for sequences where

$$E^n\{a(k)\} = a(k+n)$$

Example

$$E(2^k) = 2^{k+1}$$
$$E^2(2^k) = 2^{k+2}$$
$$E^3(2^k) = 2^{k+3}$$

Can this work for any $n \times n$ matrix?

From Thai's paper, we know that

$$A^k = \sum_{a=1}^{r} \sum_{n=0}^{m_a - 1} M_{na} \, \varphi_{n,a_r}(k)$$

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$$A^k = \sum_{n=1}^{R} M_n \, \varphi_n(k)$$

Can this work for any $n \times n$ matrix?

Knowing what the phi sequences are, we may create a system of equations with the shift operator. Setting $k = 0$, we have

$$
\begin{aligned}
A^k &= I &&= M_1\varphi_1(0) + M_2\varphi_2(0) + M_3\varphi_3(0) + \dots \\
E\{A^k\} &= A &&= M_1\varphi_1(1) + M_2\varphi_2(1) + M_3\varphi_3(1) + \dots \\
E^2\{A^k\} &= A^2 &&= M_1\varphi_1(2) + M_2\varphi_2(2) + M_3\varphi_3(2) + \dots \\
\vdots &= \vdots &&= \vdots \\
E^{R-1}\{A^k\} &= A^{R-1} &&= M_1\varphi_1(R-1) + M_2\varphi_2(R-1) + \dots
\end{aligned}
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\vdots &= \vdots &&= \vdots \\
E^{R-1}\{A^k\} &= A^{R-1} &&= M_1\varphi_1(R-1) + M_2\varphi_2(R-1) + \dots
\end{aligned}
$$

Can this work for any $n \times n$ matrix?

Representing as a matrix equation, we have

$$\begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^{R-1} \end{bmatrix} = \begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \dots & \varphi_R(0) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \dots & \varphi_R(1) \\ \varphi_1(2) & \varphi_2(2) & \varphi_3(2) & \dots & \varphi_R(2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_1(R-1) & \varphi_2(R-1) & \varphi_3(3) & \dots & \varphi_R(R-1) \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_R \end{bmatrix}$$

Where

$$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_R \end{bmatrix}$$

is our unknown matrix to solve for.

Can this work for any $n \times n$ matrix?

We will let B equal to

$$\begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \dots & \varphi_R(0) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \dots & \varphi_R(1) \\ \varphi_1(2) & \varphi_2(2) & \varphi_3(2) & \dots & \varphi_R(2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_1(R-1) & \varphi_2(R-1) & \varphi_3(3) & \dots & \varphi_R(R-1) \end{bmatrix}$$

B is a special matrix called the **Matrix of Casorati**, where the matrix is made from a set of functions and their E shift. Its determinant is called the **Casoratian** If we want to find a unique answer for our coefficient matrices, B must be invertible or the determinant of B must be non-zero.

Can this work for any $n \times n$ matrix?

Theorem

Let $u_1(t), u_2(t), \ldots, u_n(t)$ be a set of functions. If these functions are linearly independent, then the Casoratian is non-zero for all t.

Can this work for any $n \times n$ matrix?

Knowing that the phi functions are linearly independent, we know we find that the determinant of B is non-zero. Therefore, B is invertible and we may find the set of unknown coefficient matrices. Therefore, we may use Fulmer's Method for any $n \times n$ matrix.

$$B^{-1} \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^{R-1} \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_R \end{bmatrix}$$

From Davidson's class, we learned how to solve e^{At} by the Laplace Transform.

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$e^{At} = e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^t \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Now, we will show a quicker way to find e^{At} by knowing A^k from the $\mathcal{Z}$-Transform.

Find e^{At} if

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

From a previous example, we found that

$$A^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Next, we can manipulate this equation by adding a summation, multiplying by t^k, and dividing by $k!$ to both sides. We get the following:

$$\sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{k!} + \sum_{k=0}^{\infty} \frac{kt^k \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}}{k!}$$

From calculus, we know we can rewrite this as:

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{k!} + \sum_{k=0}^{\infty} \frac{kt^k \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}}{k!}$$

Since the matrices are not dependent on k we can move them out of the summation.

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sum_{k=0}^{\infty} \frac{kt^k}{k!}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!} + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sum_{k=1}^{\infty} \frac{t^k}{(k-1)!}$$

Again, from calculus we know we can rewrite the summations as a Taylor Series, giving us the following equation:

$$e^{At} = e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^t \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Summary

- Using the φ functions and the $\mathcal{Z}$-transform, we found that Fulmer's method may be used to calculate A^k.
- Using the formula for A^k, we may calculate e^{At}.