Team Session

These questions require exact numerical answers. Hand written exact answers must be written with fractions reduced, radicals simplified, and denominators rationalized. Do not make an approximation for π or other irrational numbers. Answers must be exact. Large numbers should not be multiplied out, *i.e.*, do not try to multiply out 20! or 6^{40}. The tiebreaker for the team competition is time. If your team reaches a point where you are satisfied or expect that you will not have more solutions in the allotted time, then you may wish to turn in your paper a little early to get a time advantage.

1. How many different finite sequences (of one or more terms) of consecutive positive integers have a sum of 105?

2. If n points are given in the plane and each pair are at most 1 unit apart, then the circle of smallest radius that is guaranteed to contain all the points has radius r. Find r.

3. Recall that the area of an ellipse defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab. The set of points (x, y) whose distance from the line $y = -6$ is twice the distance from (x, y) to the point $(0, -1)$ is an ellipse. Find the area of the ellipse.

4. For the purpose of this problem treat the earth as a sphere of radius 4000 miles. The latitude and longitude of New Orleans is 30° N and 90° W. On the other side of the world at 30° N and 90° E is Lhasa Tibet. If John travels from New Orleans to Lhasa along the 30° latitude line and Jane travels to Tibet along the shortest path on the surface of the earth, then how much farther does John travel?

5. Two altitudes of a triangle are of length 1 and 3. What is the largest number x so that the length of the third altitude must be larger than x?

6. Find the largest real solution to

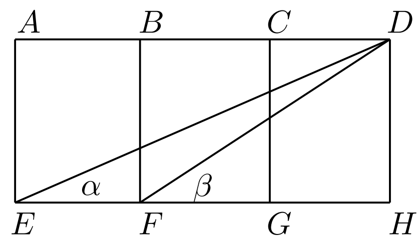
$$-6 = (x - 3)(x + 1)(x - 4)(x + 2).$$

7. A sequence begins with two unidentified integers. If the third and subsequent elements of the sequence can be computed by summing the previous two elements, and the 11th and 12th elements are 107 and 173 respectively, then what are the first two elements of the sequence?

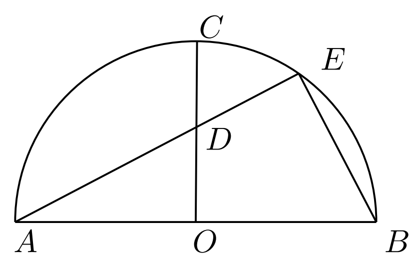
8. Suppose a square and a regular hexagon have equal areas. Find the ratio of the area of a circle inscribed in the square to the area of the circle inscribed in the hexagon.

9. This question concerns the representation of numbers in base 7. For example, recall that $4! = 24$ is represented in base 7 as 33. Suppose the number 200! is written in base 7. Starting at the right, how many zeros are encountered before reaching the first nonzero digit?

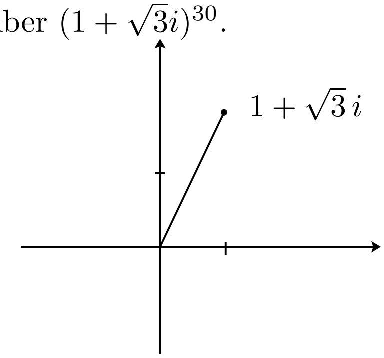
10. In the diagram, $ABFE$ $BCGF$ and $CDHG$ are squares. If the degree measure of $\angle DEF$ is α and the degree measure of $\angle DFH$ is β, then find $\alpha + \beta$.



11. The point O is the center of the semicircle $ACEB$ of radius 1. If $\angle COB$ is a right angle and $\overline{OD} : \overline{DC} = 2 : 1$, then find the length of $\overline{EB}$.



12. Compute the complex number $(1 + \sqrt{3}i)^{30}$.



13. An ice cream parlor sells a sundae special consisting of two scoops ice cream with four toppings. They have 20 toppings from which to choose, and they will allow you to choose the same topping up to four times, as long as the total number of choices is exactly four. How many different combinations of choices of four toppings are there?