2006 LSU Math Contest
Open Session

Questions 1 - 12 are worth 1 point each and questions 13 - 22 are worth 2 points each.

No calculators are allowed.

Pictures are only sketches and are not necessarily drawn to scale or proportion.

You have one hour and twenty minutes to complete the entire morning exam.

Questions 1 - 12 Multiple Choice

Please:
- *Use the answer sheet for your answers.*
- *Answer only one choice A, B, C, D, or E for each question by circling your answer on the answer sheet.*
- *Erase clearly any answer you wish to change.*
- *Do not make stray marks on the answer sheet.*

1
Let θ be an irrational number. Which of the following statements is always true

A θ^3 is irrational B $\sin\theta$ is irrational C $\sqrt{|\theta|}$ is irrational

D $\dfrac{1}{1+\pi\theta}$ is irrational E none of the above

2
Two different real numbers y and z are roots of the quadratic equation $ax^2 + c = 0$ with $a, c \neq 0$.
Consider the following statements

(i) $y + z = 0$

(ii) $y^2 + z^2 = 0$

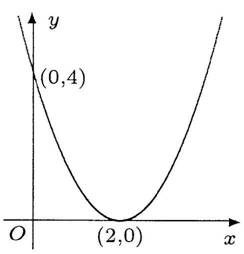
(iii) $y^3 + z^3 = 0$

(iv) $yz = 0$

How many of them are true?

A 0 B 1 C 2 D 3 E 4

3
This diagram



could be the graph of

A $y = x^2 + 4$ B $y = x^2 - 2x$ C $y = x^2 + 4x$

D $y = (x-2)^2$ E $y = (x+2)^2$

4
In the equation

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2},$$

if f and f_2 are both halved, what is the effect on f_1?

A quadrupled B doubled C unchanged

D halved E quartered

5
How many planes of symmetry does a cube have?

A 3 B 5 C 6 D 9 E 12

6
The equation $x^{x\sqrt{x}} = (x\sqrt{x})^x$ has two solutions in positive real numbers x. One obvious solution is $x = 1$. The other one is $x =$

A $\frac{1}{4}$ B $\frac{2}{3}$ C $\frac{9}{4}$ D $\frac{3}{2}$ E none of these

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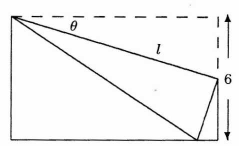
7

One side of a rectangular piece of paper is 6 in and the adjacent sides are longer than 6 in. One corner of the paper is folded so that it rests on the opposite longer side. If the length of the crease is l inches and it makes an angle θ with the position of the long side as shown, then l is



A $6\tan\theta$ B $\dfrac{3}{\sin\theta\cos^2\theta}$ C $\dfrac{6}{\sin^2\theta\cos\theta}$

D $\dfrac{3}{\sin\theta\cos\theta}$ E $\dfrac{3}{\sin^3\theta}$

8
A car dealer sold two quality used cars for \$9999 each. On one she made a 10% profit and on the other a 10% loss. What was her overall profit or loss over the two transactions?

A loss of \$202 B loss of \$101 C broke even

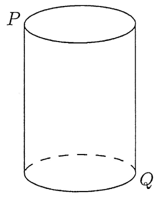
D profit of \$101 E profit of \$202

9
Let $F : \mathbb{R} \to \mathbb{R}$ be any function. About which line are the graphs of $y = F(x-1)$ and $y = F(-x+1)$ symmetric with respect to each other?

A $y = 0$ B $x = -1$ C $x = 0$ D $x = 1$ E $y = x$

10

The diagram represents a circular cylinder of circumference 6 in and height 4 in. Point P on the top rim is diametrically opposite point Q on the bottom rim. What is the shortest distance, in inches, from P to Q along the surface of the cylinder?



A $\sqrt{52}$ B $4+\dfrac{6}{\pi}$ C $\sqrt{16+\dfrac{36}{\pi^2}}$ D 5 E 7

11

If x and y are non-zero real numbers such that

$$|x|+y=3 \quad \text{and} \quad |x|y+x^3=0,$$

then the integer nearest to $x-y$ is

A -3 B -1 C 2 D 3 E 5

12

Consider the following four statements about the equation

$$x|\,x|+px+q=0.$$

(1) It has at most three roots.
(2) It has at least one real root.
(3) It has real roots only if $p^2-4q\geq 0$.
(4) It has three real roots if $p<0$ and $q>0$.

How many of them are false?

A 0 B 1 C 2 D 3 E 4

Questions 13 - 22 Exact Answer Questions

These next ten questions are exact numerical or algebraic answers. Hand written exact answers must be written on the answer sheet with fractions reduced, radicals simplified, and denominators rationalized. Do not make an approximation for π or other irrational numbers. Answers must be exact. Large numbers should not be multiplied out, i.e., do not try to multiply out $20!$ *or* 6^{40}.

13 Evaluate

$$\sqrt[3]{\frac{1}{10^{-\log_{10}1000}}}\,.$$

14 Water can be pumped into or out of a tank via three pipes A, B and C. Pipe A can fill the tank in 4 hours. Pipe B can fill the tank in 6 hours. Pipe C can *empty* the tank in 5 hours. If the water tank is empty and all three pipes begin operating at the same time, how long will it take to fill the tank?

15 A solid square-base pyramid, with all edges of unit length, and a solid triangle-base pyramid (tetrahedron), also with all edges of unit length, are glued together by matching two triangular faces.

How many faces does the resulting solid have?

16 Let x, y and z be real numbers such that $3x$, $4y$ and $5z$ form a geometric progression while $\frac{1}{x}$, $\frac{1}{y}$ and $\frac{1}{z}$ form an arithmetic progression. What is the value of

$$\frac{x}{z}+\frac{z}{x}\ ?$$

17 Simplify the expression

$$0.1\left[2(3x-y)^2-3(x-2y)^2-10y^2\right]-1.5x^2.$$

18 Find the ratio of the areas of two regular polygons of n sides, inscribed in and circumscribed about a given circle.

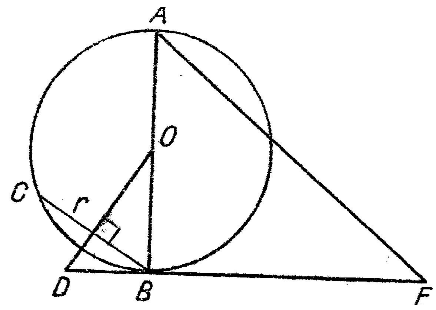
19 Compute explicitly the positive number n such that $n^{\log_{19}89}=89^2$.

20 A rectangle's sides lengths are natural numbers. Find all possible dimensions of the rectangle if the number expressing its area is the same as the number expressing its perimeter.

21 A *palindrome number* is a positive integer that reads backwards the same as it reads forwards, e.g. 1234321.

Let X be the set of all 13-digit positive integers. An integer is chosen at random from X. The probability that it is a palindrome number is $\frac{1}{10^n}$. Find n.

22 In the picture below $CB=r=AO$, the radius of the circle. $\overline{DF}$ is perpendicular to $\overline{AB}$, $\overline{OD}$ is perpendicular to $\overline{CB}$, $DF=3r$.



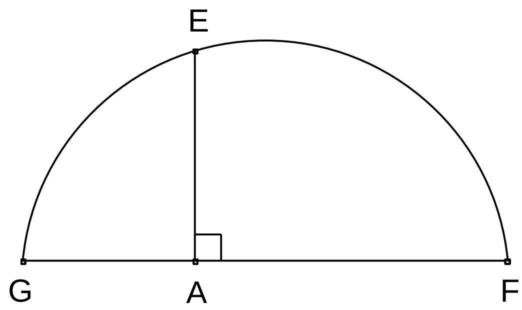
Find the exact length of $\overline{AF}$.

[The above XVIIth century geometrical construction (of "flattening" of the circle) determines the segment $\overline{AF}$ the length of which is approximately $r\pi$.]

Tie Breaker

This tie breaker question [$\boldsymbol{TQ}$] is graded as an essay question i.e., it is graded for the clarity of explanation and argument as well as correctness.
It is the only question graded for partial credit.
It is graded only to separate first, second, and third place ties.

Please give a detailed explanation on the answer sheet to the following two part question.



TQa Find the length of the segment $\overline{AE}$ in terms of $AF = r\pi$ and $GA = r$ as shown on the diagram.

TQb The construction of Question **22** above, if it gave an accurate value of $r\pi$, combined with the classical construction of the segment $\overline{AE}$ here on the right, would provide a solution to the famous problem of antiquity of squaring the circle, i.e. *of constructing a square of an area equal to that of the circle by a method which involves the use only of the compass and of the ruler as a single straight-edge.* Explain how.