2009 LSU Math Contest
Open Session

Questions 1 - 12 are worth 1 point each and questions 13 - 24 are worth 2 points each.

No calculators are allowed.

Pictures are only sketches and are not necessarily drawn to scale or proportion.

You have one hour and twenty minutes to complete the entire morning exam.

Questions 1 - 12 Multiple Choice

Please:
- *Use the answer sheet for your answers.*
- *Answer only one choice A, B, C, D, or E for each question by circling your answer on the answer sheet.*
- *Completely erase any answer you wish to change.*
- *Do not make stray marks on the answer sheet.*

1

Let a, b, and c be positive real numbers. If $\frac{a}{b} < \frac{a+c}{b+c}$, then

A $b^2 < a$ B $b < a$ C $b = a$ D $b > a$ E $a^2 > b$

2

The values of the function $f(x) = \sin x - \cos x$, $0 \le x \le 2\pi$, form the closed interval

A $[-2, 2]$ B $[-\sqrt{2}, \sqrt{2}]$ C $[-1, 1]$ D $[0, 1]$ E $[-1, 0]$

3

Let $f(x) = a(x - x_1)(x - x_2)$ where a, x_1, and x_2 are real numbers such that $a \neq 0$ and $x_1 + x_2 = 0$. Then

A $f(x)$ is monotone on $(-\infty, 0)$ and on $(0, +\infty)$

B $f(x)$ has a minimum at $x = 0$

C $f(x)$ has a maximum at $x = 0$

D $f(x)$ is increasing on $(0, +\infty)$

E None of the above is correct.

4

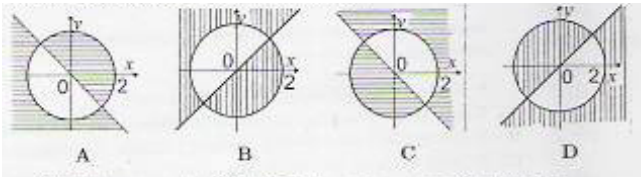
Let $S = \{(x, y) \mid x^2 + 2xy + y^2 - 3x - 3y + 2 = 0\}$, then S

A is a circle B is a line C consists of two lines

D has only one element E is empty

5

Let $S = \{(x, y) \mid (x^2 + y^2 - 4)(x + y) < 0\}$ be a subset of $x, y-$plane. Which picture's shaded part portrays S ?



E None of the above.

6

If in a cube the distance from one corner of the cube to the farthest corner is $\sqrt[3]{3}$. then the distance from one corner of its face to the opposite corner of the same face is

A $\sqrt{2}$ B $\sqrt[3]{2}$ C $\sqrt[6]{\frac{8}{3}}$ D $\sqrt{\frac{2}{3}}$ E $\sqrt{3}$

7

Triangle ABC has area 4π. If the points P, Q, and R are midpoints of the sides BC, AC, and AB respectively then the area of the triangle PQR is

A $2\sqrt{\pi}$ B $\sqrt{\pi}$ C π D 2π E $\pi/2$

8

Which of the following regions in a plane *cannot contain* a circle of circumference 12 ?

A a disk of radius 13

B the exterior of a disk of radius 13

C a square of area 16

D an equilateral triangle with perimeter 13

E None of these

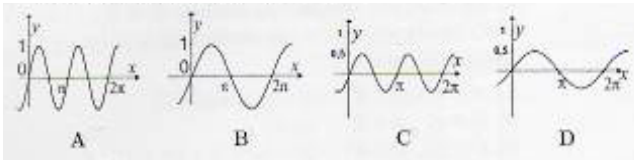
9

Recall that $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$. The maximum value of the integer x such that 3^x divides 30! is:

A 30 B 14 C 13 D 10 E 4

10

Which of the following is a graph of $f(x) = \frac{1}{2}\sin(2x)$



E None of the above.

11
Consider the polynomial $P(x) = x^{2n+1} + x^n - 1$ with $n \geq 1$. Which of the following statements is true:

A $P(x)$ is a product of two polynomials of even degree

B $P(x)$ has integer roots

C some of the roots of $P(x)$ are rational numbers

D $P(x)$ has a root which is an irrational number

E None of these is true

12
Which of the following statements about the equation

$$2^{-|x|} = x^2 + 1$$

is true?

A it has exactly two real solutions

B it has no real solutions

C 1 is a solution

D -1 is a solution

E None of the above is true

Questions 13 - 24 Exact Answers

These next twelve questions require exact numerical or algebraic answers. Hand written exact answers must be written on the answer sheet with fractions reduced, radicals simplified, and denominators rationalized (Improper fractions can be left alone or changed to mixed fractions). Do not make an approximation for π or other irrational numbers. Answers must be exact. Large numbers should not be multiplied out, i.e., do not try to multiply out $20!$ *or* 6^{40}.

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13 Determine all pairs (a, b) of real numbers that satisfy the system of equations

$$a + \log a = b,$$
$$b + \log b = a.$$

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14 What is the value of the expression

$$\frac{1}{\log_2 100!} + \frac{1}{\log_3 100!} + \frac{1}{\log_4 100!} + \cdots + \frac{1}{\log_{100} 100!}?$$

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15 Find $\arcsin x + \arccos x$ if $-1 \leq x \leq 1$.

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16 The function f satisfies $f(1 + x) = f(1 - x)$ for all real numbers x. Moreover, $f(x) = 0$ has exactly four distinct real roots. What is the sum of these roots ?

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17 The polynomial in x

$$P(x) = 1 - x + x^2 - x^3 + \cdots - x^{19} + x^{20}$$

is expressed as a polynomial in y

$$Q(y) = a_0 + a_1 y + a_2 y^2 + \cdots + a_{19} y^{19} + a_{20} y^{20},$$

where $y = x - 4$. Find the number $a_0 + a_1 + \cdots + a_{20}$.

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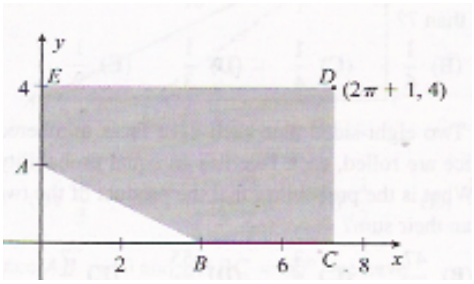
18 A point P is selected at random from the interior of the pentagon with vertices $A = (0, 2)$, $B = (4, 0)$, $C = (2\pi + 1, 0)$, $D = (2\pi + 1, 4)$, and $E = (0, 4)$. What is the probability that $\angle APB$ is *obtuse*?



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19 Let S be the set of permutations of the numbers 1, 2, 3, 4, 5, 6 for which the first term of the permutation is not 1. A permutation is chosen randomly from S. Find the probability that the second term of the chosen permutation is 2.

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20 Find the coefficient of x^{99} in the polynomial expansion of $(x - 1)(x - 2) \ldots (x - 100)$.

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21 Four spheres each of radius 10in lie on a horizontal table so that the centers of the spheres form a square of side 20in. A fifth sphere of radius 10in is placed on them so that it touches each of the spheres without disturbing them. How many inches above the table is the center of the fifth sphere?

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22 How many triangles are there such that each side has integral length and the longest side has length 11?

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23 Let x, y and z be real numbers such that $3x$, $4y$ and $5z$ form a geometric progression while $\frac{1}{x}$, $\frac{1}{y}$ and $\frac{1}{z}$ form an arithmetic progression. What is the value of

$$\frac{x}{z} + \frac{z}{x}\,?$$

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24 There are eleven positive integers n such that there exists a convex polygon with n sides whose angles, in degrees, are unequal integers that are in arithmetic progression. Find the sum of these values of n.

Tie Breaker requiring Full Solution

Please give a **detailed explanation** on the answer sheet **of** your **solution to Question 24**.

This tie breaker question is graded as an essay question, i.e. it is graded for the clarity of explanation and argument as well as correctness.

It is the only question graded for partial credit. Do not hesitate to write your thoughts even if your solution is not rigorous!

It is graded only to separate first, second, and third place ties.