

Additive Schwarz & Multilevel Methods

Jung-Han Kimn
Department of Mathematics
Louisiana State University

ABSTRACT. This handout will explain the fundamental ideas of additive Schwarz and multilevel preconditioning strategies for scalable parallel numerical solvers for large-scale linear systems.

1. Introduction

The development of scalable parallel numerical methods for large algebraic systems is central in solving very large scale linear systems. The most widely used methods for such problems are Krylov subspace methods such as GMRES and the Conjugate Gradient method.

The key to scalability is preconditioning to accelerate the convergence of the iterative solver. Preconditioning means replacing the system $Ax = b$ to a system that is more easily solved. For example, one might replace $Ax = b$ to $M^{-1}Ax = M^{-1}b$ where M is an approximation to A with the properties that

1. $M^{-1}A$ is well conditioned or has few extreme eigenvalues,
2. $Mx = b$ is easy to solve.

A careful choice of M can often make the condition number of $M^{-1}A$ much smaller than the condition number of A and thus accelerate convergence.

Domain decomposition (DD) methods provide a very natural way of deriving parallel algorithms for the numerical solution of linear systems and can often be viewed as preconditioners for iterative methods. When DD algorithms are used, a large number of subproblems can be solved in parallel. The local interaction is through the exchange of information between neighboring subdomains. It also necessary to introduce a global coarse grid, with a few gridpoints per subdomain,

to provide global interaction between the subregions and to obtain fast convergence in case of many subdomains.

2. Overlapping Schwarz methods

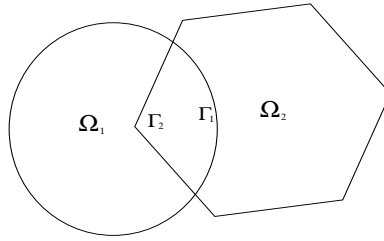


FIGURE 1. An example of two overlapping subdomains with two artificial interfaces.

Consider the domain Ω , as shown in Figure 1, with two overlapping subdomains $\Omega_1 \cup \Omega_2$ on which we wish to solve the linear PDE

$$\begin{aligned} Lu &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Let $\partial\Omega$ denote the boundary of Ω and the artificial boundaries, Γ_i , are the part of the boundary of Ω_i that is interior to Ω .

The earliest known domain decomposition method is the alternating method of H. Schwarz dating back to 1870. For the domain with $\Omega = \Omega_1 \cup \Omega_2$, the iteration method begins by selecting an initial guess u_2^0 for the values in Γ_1 , and we solve the boundary value problem iteratively for u_1^n ,

$$\begin{aligned} Lu_1^n &= f \quad \text{in } \Omega_1 \\ u_1^n &= g \quad \text{on } \partial\Omega_1 \setminus \Gamma_1 \\ u_1^n &= u_2^{n-1} \quad \text{on } \Gamma_1. \end{aligned}$$

Then we solve the problem for u_2^n ,

$$\begin{aligned} Lu_2^n &= f \quad \text{in } \Omega_2 \\ u_2^n &= g \quad \text{on } \partial\Omega_2 \setminus \Gamma_2 \\ u_2^n &= u_1^n \quad \text{on } \Gamma_2. \end{aligned}$$

In many applications, it is possible to use a matching grid in the overlap region to avoid the duplication of the unknowns on the overlap. The matching version of the alternating method is known as the **multiplicative Schwarz** method. Writing the linear system for the

discretized problem as $Au = f$, we can write the iteration in two fractional steps:

$$u^{n+1/2} = u^n + \begin{pmatrix} A_{\Omega_1}^{-1} & 0 \\ 0 & 0 \end{pmatrix} (f - Au^n),$$

and

$$u^{n+1} = u^{n+1/2} + \begin{pmatrix} 0 & 0 \\ 0 & A_{\Omega_2}^{-1} \end{pmatrix} (f - Au^{n+1/2}),$$

where the A_{Ω_i} is the discrete form of the operator L , restricted to Ω_i . Since each iteration involves sequential fractional steps, this is not ideal for parallel computing.

3. Additive Schwarz methods

The **additive Schwarz** method (ASM) can be considered as a parallelizable version of the multiplicative Schwarz method and can be written as

$$u^{n+1} = u^n + \left[\begin{pmatrix} A_{\Omega_1}^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A_{\Omega_2}^{-1} \end{pmatrix} \right] (f - Au^n).$$

This can be written as

$$u^{n+1} = u^n + (B_1 + B_2)(f - Au^n) \quad \text{with} \quad B_i = R_i^T A_{\Omega_i}^{-1} R_i,$$

where R_i is the rectangular restriction matrix that returns the vector of components defined in the interior of Ω_i .

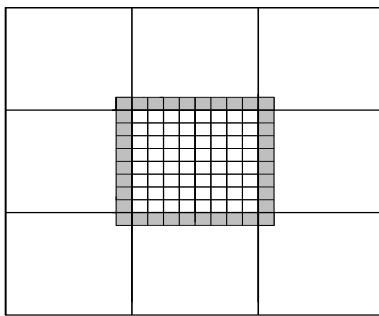


FIGURE 2. A patch with overlap in eight neighboring patches.

The idea of ASM carries over immediately to methods that involve more than two subdomains. For a domain $\Omega = \cup_i \Omega_i$, ASM can be written as

$$u^{n+1} = u^n + \sum_i B_i (f - Au^n) \quad \text{with} \quad B_i = R_i^T A_{\Omega_i}^{-1} R_i. \quad (2)$$

The additive Schwarz methods may be viewed as generalizations of block Jacobi methods.

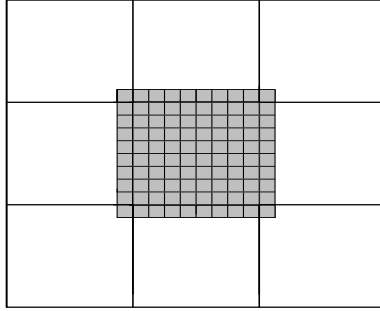


FIGURE 3. The restriction operator R_i of ASM.

4. Restricted additive Schwarz methods

ASM is available in several large parallel software libraries such as PETSc. The restricted additive Schwarz method (RAS) was introduced recently by Cai & Sarkis. RAS is the default parallel preconditioner in PETSc, it saves substantial communication costs and requires fewer iterations for convergence than ASM for many applications.

RAS was constructed from ASM with very simple changes. The iteration can be induced from (2) and written as

$$u^{n+1} = u^n + \sum_i \tilde{B}_i^0 (f - Au^n) \quad \text{with} \quad \tilde{B}_i^0 = (R_i^0)^T A_{\Omega_i}^{-1} R_i^T, \quad (3)$$

where R_i^0 is restriction to the non-overlapping part of Ω_i . Two illustrations of B_i and \tilde{B}_i^0 are given in Figure 3 and Figure 4 respectively. During iterations of RAS, residuals from ghost values are used but computed values in the ghost region are discarded.

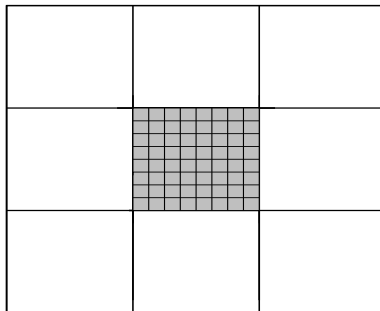
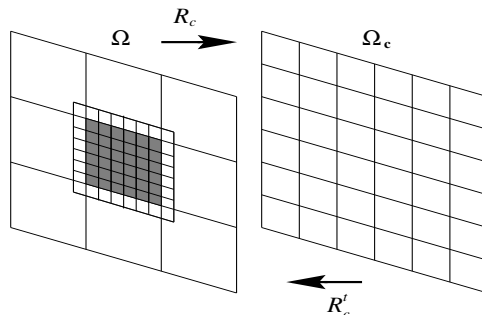


FIGURE 4. The restriction operator R_i^0 of RAS.

FIGURE 5. The relation between Ω and Ω_c

5. Multilevel methods

Single level methods are most effective only for a small number of subdomains. The problem with single level methods is that the information about given f and g in (1) in one subdomain is conveyed through to other subdomains only by passing through all the intermediate subdomains. The iterative linear system solver needs some mechanism for the global communication of information at each iteration. The most commonly used mechanism for transmitting global information is the **multilevel** method. A multilevel method uses one or more coarse spaces Ω_c and solves an appropriate problem on each coarse grid.

Good parallel linear solvers for very large scale linear systems should be **scalable**, that is, their performance should be insensitive to the number of subdomains. The convergence rate of the single level ASM deteriorates when the number of subdomains becomes large. This weakness has been overcome by introducing a coarse global problem set over the whole domain in order to provide a global communication mechanism between all subdomains. The overall iteration of two-level ASM can be written as

$$\begin{array}{ll}
 r^n & := b - Au^n & \text{Compute the residual on } \Omega \\
 r_c & := R_c r^n & \text{Restrict the residual to } \Omega_c \\
 A_c c_c & := r_c & \text{Solve on } \Omega_c \text{ (} A_c = R_c A R_c^t \text{)} \\
 c^n & := R_c^t c_c & \text{Interpolate the correction} \\
 u^n & := u^n + c^n & \text{Compute the new approximation} \\
 u^{n+1} & := u^n + \sum_i B_i r^n & \text{ASM on } \Omega
 \end{array}$$