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# *Optimal Fiber Configurations for Maximum Torsional Rigidity*

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## Contents



## 1. Introduction

B. DE SAINT-VENANT [13] proposed in 1856 that among all prismatic shafts with given cross-sectional area that the greatest torsional rigidity is obtained by a shaft with circular cross-section. This proposition was proved by G. POLYA [11], who, with A. WEINSTEIN, extended this result to multiply connected cross sections [12]. In this treatment we consider prismatic shafts with given cross-sectional area reinforced with fibers of greater stiffness. We suppose that the fibers are imperfectly bonded to the shaft and that the joint area of fiber cross sections is fixed. Subject to these conditions we investigate the problem of finding the shaft and fiber cross sections that yield the maximum torsional rigidity. The answer is shown to depend upon the magnitude of a distinguished parameter "$R_{cr}$". This parameter has dimensions of length and measures the influence of the imperfect bond. Before stating the results we provide the necessary background and a description of the parameter $R_{cr}$.

Imperfect bonding or partial adhesion between matrix and fiber is often caused by interfacial damage due to service or as a consequence of the surface properties of the fiber and matrix materials. Imperfect bonds are characterized by the loss of continuity in the displacement across the fiber-matrix interface. In this treatment partial adhesion is modeled by an interfacial surface across which the tangential components of the displacement are discontinuous. The traction is assumed continuous across the interface and the relative tangential displacement is proportional to the tangential traction. No interpenetration between matrix and fiber is allowed and the normal component of the displacement is continuous across the interface. The stiffness of the interface is characterized by the parameter $\alpha$, relating the tangential traction to the relative tangential displacement. This parameter has dimensions of shear stiffness per unit length and ranges between zero and infinity. The limiting case $\alpha = \infty$, corresponds to perfect bonding for which the displacement is continuous across the interface. The case of no adhesion along directions parallel to the interface is captured in the limit $\alpha = 0$. This interface model was used by LENE & LEGUILLON [4] in their treatment of the softening of effective moduli arising from damage. Flexible interface models similar to the type treated here can be found in the work of JONES & WHITTER [8]. A comprehensive treatment of interface models as they relate to imperfect bonding is provided in the recent book of ABOUDI [1].

We suppose that the shaft is a cylinder of constant cross section of length $h$ with generators parallel to the $x_3$ axis. The cylinder cross section is denoted by $\Omega$ and is a simply connected domain with Lipschitz continuous boundary in the $(x_1, x_2)$-plane. We suppose that all fibers run the length of the shaft and that each fiber is a cylinder of constant cross section with generators parallel to the $x_3$ axis. The boundary of each fiber is assumed to be Lipschitz continuous. Both matrix and fibers are assumed to be made from linearly elastic isotropic materials. The shear moduli of the fibers and matrix are denoted by $\mathrm{G_f}$ and $\mathrm{G_m}$ respectively. The fibers are assumed to provide reinforcement and so we set $\mathrm{G_f} > \mathrm{G_m}$. The elastic deformation inside the shaft is given by the displacement field, $\boldsymbol{u} = (u_1, u_2, u_3)$ and the associated $3 \times 3$ stress tensor is denoted by $\sigma_{ij}$. We fix coordinates so that the base of the shaft lies on the $x_3 = 0$ plane and the $x_3$ axis lies within the shaft. The sides of the shaft are kept traction-free and we fix $u_1 = u_2 = 0$ and $\sigma_{33} = 0$ on the base of the shaft. The shaft is subjected to a twist of angle $\theta$ per unit length. At $x_3 = h$ we have $u_1 = -\theta h x_2$, $u_2 = \theta h x_1$, with $\sigma_{33} = 0$. We denote a configuration of $N$ fibers with cross sections $\Sigma_i$, $i = 1, \dots, N$ by $A$. Here, $A \subset \Omega$ and $A = \cup_{i=1}^{i=N} \Sigma_i$. The torsional rigidity is the ratio between the resultant torsional moment over the cross section $\Omega$ and the twist per unit length $\theta$. Denoting the torsional rigidity by $\mathrm{T}(A, \Omega, \alpha)$ we have:

$$\theta \mathrm{T}(A, \Omega, \alpha) = \int_\Omega (x_1 \sigma_{23} - x_2 \sigma_{13})\, dx. \tag{1.1}$$

Here $dx = dx_1 dx_2$.

We naturally expect that the torsional rigidity of an imperfectly bonded fiber-reinforced shaft is less than that of a perfectly bonded shaft. However, equality holds for a circular shaft reinforced with a single centered fiber with circular cross-section: see Section 4. Indeed, when the shaft radius is $R$ and fiber radius is $a$, the torsional rigidity is given by

$$\tfrac{1}{2}\pi \mathrm{G_m}(R^4 - a^4) + \tfrac{1}{2}\pi \mathrm{G_f} a^4; \tag{1.2}$$

this is computed in Section 4.

Formula (1.2) shows that the torsional rigidity for this configuration is independent of the tangential interfacial stiffness $\alpha$. This is due to the fact that the traction vanishes at the matrix-fiber interface; see Section 4. We find that the rigidity given by (1.2) holds even for the extreme case when there is a complete loss of adhesion in the direction tangential to the interface; see Section 7. We see from (1.2) that the rigidity for the concentric circular fiber-shaft configuration is strictly increasing with the fiber radius.

To understand how imperfect bonding influences the torsional rigidity, we define the parameter

$$\mathrm{R_{cr}} = \frac{\alpha^{-1}}{\mathrm{G_m}^{-1} - \mathrm{G_f}^{-1}}. \tag{1.3}$$

This parameter has dimensions of length and is a measure of the relative influence between the interfacial compliance and the mismatch between matrix and fiber compliance. We show that $\mathrm{R_{cr}}$ sets the length scale at which the effects of partial interfacial adhesion spoil the effects of a stiff reinforcement.

The first result that we present applies to shafts with circular cross section reinforced with a single fiber with circular cross section. For a shaft of radius $R$ we choose coordinates so that the shaft cross section is a disk centered at the origin. We denote this cross section by $D_0(R)$. The fiber cross section is a disk of radius $a$ with center denoted by $\check{x}$ and we denote the fiber cross section by $D_{\check{x}}(a)$. The torsional rigidity for this configuration is written

$$\mathrm{T}(D_{\check{x}}(a), D_0(R), \alpha), \tag{1.4}$$

and we have

**Theorem 1.1.** *If the fiber cross-sectional radius is less than or equal to* $\mathrm{R_{cr}}$, *then the maximum torsional rigidity is obtained by centering the fiber inside the shaft; otherwise the maximum torsional rigidity is given by moving the fiber "off-center", i.e., if* $a \leqq \mathrm{R_{cr}}$, *then*

$$\mathrm{T}(D_{\check{x}}(a), D_0(R), \alpha) \leqq \tfrac{1}{2}\pi \mathrm{G_m}(R^4 - a^4) + \tfrac{1}{2}\pi \mathrm{G_f} a^4 \tag{1.5}$$

*and if* $a > \mathrm{R_{cr}}$, *then*

$$\mathrm{T}(D_{\check{x}}(a), D_0(R), \alpha) > \tfrac{1}{2}\pi \mathrm{G_m}(R^4 - a^4) + \tfrac{1}{2}\pi \mathrm{G_f} a^4 \quad \textit{for} \quad \check{x} \neq 0. \tag{1.6}$$

This result is proved in Section 5. We remark that (1.6) holds in the perfect bonding limit $\alpha = \infty$; see Section 7. To understand the physical significance of Theorem 1.1 we consider a single fiber reinforcement of circular cross section and allow the parameter $\mathrm{R_{cr}}$ to change. Indeed, (1.5) shows that, for sufficiently large $\mathrm{R_{cr}}$, the centered fiber configuration is optimal. For this case we see that the interfacial compliance is large with respect to the relative compliance between the matrix and fiber. Therefore the most rigid design is the centered fiber configuration where the interface does not support any traction. When $R_{cr}$ is sufficiently small, the effect of the compliant interface is reduced and inequality (1.6) shows that the fiber is best utilized by placing it off-center.

Next we consider configurations that correspond to a shaft of circular cross section of radius $R$, reinforced with $N$ identical fibers of circular cross section. The common radii of the fibers is denoted by $a$ and the center of the $i^{\text{th}}$ fiber is denoted by $x_i$, $1, \ldots, N$. The cross section of the $i^{\text{th}}$ fiber is a disk of radius $a$ and is denoted by $D_{x_i}(a)$. We examine the situation when the common fiber radius is precisely $R_{cr}$.

**Theorem 1.2.** *If the common cross-sectional radius of the N fibers is equal to* $R_{cr}$, *then the torsional rigidity is precisely that of a single centered circular fiber of radius* $N^{1/4}R_{cr}$ *centered at the origin. This result is independent of the location of the N fibers, i.e.,*

$$T(\cup_{i=1}^{N} D_{x_i}(R_{cr}), D_0(R), \alpha) = \tfrac{1}{2}\pi G_m(R^4 - NR_{cr}^4) + \tfrac{1}{2}\pi G_f N R_{cr}^4. \tag{1.7}$$

Here $T(\cup_{i=1}^{N} D_{x_i}(R_{cr}), D_0(R), \alpha)$ denotes the torsional rigidity of $N$ identical fibers with common cross-sectional radius $R_{cr}$. For $N$ identical fibers with common cross section $a$ we have

**Theorem 1.3.** *If* $a \leqq R_{cr}$, *then the torsional rigidity is less than or equal to the rigidity associated with a single centered circular fiber of radius* $N^{1/4}R_{cr}$, *and if* $a > R_{cr}$, *the torsional rigidity is strictly larger. This result is independent of the location of the N fibers, i.e., if* $a \leqq R_{cr}$,

$$T(\cup_{i=1}^{N} D_{x_i}(a), D_0(R), \alpha) \leqq \tfrac{1}{2}\pi G_m(R^4 - NR_{cr}^4) + \tfrac{1}{2}\pi G_f N R_{cr}^4; \tag{1.8}$$

*otherwise,*

$$T(\cup_{i=1}^{N} D_{x_i}(a), D_0(R), \alpha) > \tfrac{1}{2}\pi G_m(R^4 - NR_{cr}^4) + \tfrac{1}{2}\pi G_f N R_{cr}^4. \tag{1.9}$$

We address the more general case of shafts with arbitrary cross section reinforced with at most $N$ fibers, each with circular cross-section. Here we assume that the cross-sectional radius of each fiber may be different. We consider all shafts with given cross-sectional area and prescribed joint cross-sectional area of fibers. From this class we seek the shaft cross section and fiber configuration that produce the maximum torsional rigidity. The answer to this question is shown to depend upon the parameter $R_{cr}$. We state

**Theorem 1.4.** *Consider all shafts with cross-sectional area* $\pi R^2$ *reinforced with at most N circular fibers with prescribed joint area of fiber cross sections. We suppose that the joint area of fiber cross sections lies below* $\pi R_{cr}^2$. *Of all such fiber-reinforced shafts, the shaft with circular cross section reinforced with a single centered fiber of circular cross section has the maximum torsional rigidity.*

When the joint area of fiber cross sections lies above $\pi R_{cr}^2$, it is evident from Theorem 1.1 that a single circular fiber placed off-center in a circular shaft does better than the centered fiber configuration. Thus no assertion of the type given in Theorem 1.4 can be made when the joint cross-sectional area of fibers lies above $\pi R_{cr}^2$.

We may extend our analysis to include a larger class of admissible fiber cross sections. We allow the fiber cross section to be any convex shape with analytic boundary provided that the maximum radius of curvature on the boundary lies below $R_{cr}$.

**Theorem 1.5.** *Consider all shafts with cross-sectional area $\pi R^2$ reinforced with at most N fibers with prescribed joint area of fiber cross sections. The fiber cross sections are assumed to be convex with analytic boundary. Suppose that the joint area of fiber cross sections lies below $\pi R_{cr}^2$ and that the maximum radius of curvature on the boundary of each fiber cross section lies below $R_{cr}$. Of all such fiber-reinforced shafts, the shaft with circular cross section reinforced with a single centered fiber of circular cross section has the maximum torsional rigidity.*

Theorems 1.4 and 1.5 are consequences of an isoperimetric inequality that is derived in this paper. Before giving the isoperimetric inequality, we identify a quantity intrinsic to the fiber cross-section referred to here as the *surface traction to bulk stress quotient.* It provides a measure of the magnitude of the in-plane stress generated in the fiber cross section due to a prescribed traction on the fiber boundary parallel to the generators of the fiber. We denote a fiber cross section by $\Sigma$ and its boundary by $\partial\Sigma$. The traction on the fiber boundary parallel to the generators of the fiber is written as $(0, 0, g_3)$ and we suppose further that $g_3 = g_3(x_1, x_2)$ with $\int_{\partial\Sigma} g_3 dl = 0$. Here $dl$ is the element of arc length along the boundary. The fiber is assumed to be linearly elastic with unit shear stiffness. On the top and bottom faces of the fiber we set the in-plane displacements and normal traction to zero. The resulting stress inside the fiber is denoted by $\tau_{ij}$, and $\partial_j\tau_{ij} = 0$ in the fiber. On the fiber boundary parallel to the generators, $n$ denotes the outward directed unit normal, $\tau_{1j}n_j = \tau_{2j}n_j = 0$ and $\tau_{3j}n_j = g_3$. Solution of the traction boundary-value problem shows that the stress tensor depends only on the $(x_1, x_2)$ coordinates and the only non-zero components of the stress tensor are $\tau_{31}$ and $\tau_{32}$. The *ratio of surface traction to bulk stress* for the fiber cross section is given by $\beta_1 = \min\{\int_{\partial\Sigma}(g_3)^2 dl/(\int_\Sigma(\tau_{31}^2 + \tau_{32}^2)dx)\}$, where $dx = dx_1dx_2$. Here the minimum is taken over all tractions $g_3$ such that $\int_{\partial\Sigma} g_3 dl = 0$. For our purposes we find it convenient to express $\beta_1$ in terms of stress potentials. For a given in-plane stress $(\tau_{31}, \tau_{32})$, we introduce the harmonic function $\varphi$ for which $(\partial_2\varphi, -\partial_1\varphi) = (\tau_{31}, \tau_{32})$. Direct substitution shows that

$$\beta_1 = \min_{\varphi\in\mathscr{U}} \frac{\int_{\partial\Sigma} |\partial_\tau\varphi|^2 dl}{\int_\Sigma |\nabla\varphi|^2 dx}, \tag{1.10}$$

where $\partial_\tau$ indicates tangential differentiation along the boundary of the fiber cross section. The admissible class of trials $\mathscr{U}$ is given by

$$\mathscr{U} = \{\varphi|\,\varphi \text{ is harmonic in } \Sigma, \text{the trace of } \varphi \text{ on } \partial\Sigma \text{ lies in } H^1(\partial\Sigma),\ \textstyle\int_{\partial\Sigma}\varphi dl = 0\}.$$

Here $H^1(\partial\Sigma)$ is the space of all functions defined on $\partial\Sigma$ that are square-integrable and have square-integrable tangential derivitaves on $\partial\Sigma$, i.e., $\int_{\partial\Sigma}\varphi^2 dl \leqq \infty$ and $\int_{\partial\Sigma}|\partial_\tau\varphi|^2 dl \leqq \infty$. From its definition we see that $\beta_1$ is the largest constant $C$ for which the inequality

$$\int_{\partial\Sigma}|\partial_\tau\varphi|^2 dl \geqq C\int_{\partial\Sigma}|\nabla\varphi|^2 dx \tag{1.11}$$

holds for all $\varphi$ in the space $\mathscr{U}$.

This quantity (1.10) was introduced in the study of DC electrical conductivity properties of particulate composites; see LIPTON [7]. In that context $\beta_1$ is referred to as the "ratio of surface to volume dissipation." In two dimensions the ratio of surface traction to bulk stress is identical to the second Stekloff eigenvalue associated with the fiber cross section. (This is shown in Section 3.) For a fiber with circular cross section of radius $a$ the ratio of surface traction to bulk stress is precisely $a^{-1}$.

We now present an isoperimetric inequality for a large class of fiber-reinforced shafts.

**Theorem 1.6** (Isoperimetric inequality). *Consider all shafts with cross-sectional area $\pi R^2$, reinforced with at most N fibers. The fibers have simply connected cross sections. Suppose that the ratio of surface traction to bulk stress of each fiber cross section lies above $\mathrm{R}_{\mathrm{cr}}^{-1}$ and that the joint area of fiber cross sections lies below $\pi\mathrm{R}_{\mathrm{cr}}^2$. Of all such fiber-reinforced shafts with given cross-sectional area and prescribed joint area of fiber cross sections, the shaft with circular cross section reinforced with a concentric fiber of circular cross section has the maximal torsional rigidity, i.e.,*

$$\mathrm{T}(A,\Omega,\alpha) \leqq \tfrac{1}{2}\pi\mathrm{G_m}(R^4 - a^4) + \tfrac{1}{2}\pi\mathrm{G_f}a^4. \tag{1.12}$$

Here $\pi a^2$ is the joint area of the fiber cross sections.

We see that Theorem 1.6 is in the same spirit as the well-known proposition of SAINT-VENANT for prismatic shafts made from homogeneous isotropic elastic material. The proposition of SAINT- VENANT was established using Steiner symmetrization by G. POLYA [11] in 1948.

Next we present a tight upper bound on the torsional rigidity. We consider any shaft with cross-sectional area $\pi R^2$, reinforced with at most $N$ fibers. The fibers have simply connected cross sections, $\Sigma_1, \Sigma_2, \ldots, \Sigma_N$ and the area occupied by the $i^{\text{th}}$ fiber cross section is denoted by $|\Sigma_i|$.

**Theorem 1.7** (Optimal design). *If the ratio of surface traction to bulk stress of each fiber cross section lies above $\mathrm{R}_{\mathrm{cr}}^{-1}$, then the torsional rigidity is less than or equal to the torsional rigidity of a concentric circular fiber-shaft configuration where the radius a of the circular fiber is given by*

$$\pi a^2 = \sqrt{|\Sigma_1|^2 + |\Sigma_2|^2 + \cdots + |\Sigma_N|^2}. \tag{1.13}$$

It is evident from (1.13) and the inequality $\sqrt{|\Sigma_1|^2 + |\Sigma_2|^2 + \cdots + |\Sigma_N|^2} \leqq |\Sigma_1| + |\Sigma_2| + \cdots + |\Sigma_N|$ that the cross-sectional area of the circular fiber is less than or equal to the joint cross-sectional area of the fiber configuration. This theorem represents the extension of inequality (1.8) of Theorem 1.3 to fiber cross sections of general shape.

We consider the extreme case $\alpha = 0$, which corresponds to loss of adhesion in the direction tangential to the fibers. In this case we have $\mathrm{R}_{\mathrm{cr}}^{-1} = 0$ and we expect an isoperimetric inequality of the kind given in Theorem 1.6 to apply to any fiber-reinforced shaft. This is shown in Theorem 7.1. On the other hand, we do not have an isoperimetric inequality of the type given in Theorem 1.6 for the perfectly bonded case, i.e., $\alpha = \infty$. To see this, we consider a shaft with circular cross section reinforced with a single fiber with circular cross section. When the fiber radius lies above $\mathrm{R}_{\mathrm{cr}}$ we recall that the centered fiber configuration is suboptimal. Therefore as we pass to the extreme case $\alpha = \infty$, we find that $\mathrm{R}_{\mathrm{cr}} = 0$ and we expect the centered circular fiber in a circular shaft to be suboptimal for every choice of fiber radius. This is shown rigorously in Section 7 by passing to the limit $\alpha = \infty$ in inequality (5.1) of Theorem 5.1. Similar remarks hold for Theorem 1.7 and are established in Section 7.

The results of this analysis apply when there is partial adhesion in directions both normal and tangential to the interface. For this case we consider an imperfect interface model similar to the type introduced in the work of J. D. ACHENBACH & H. ZHU [2]. In addition to a discontinuity in the tangential displacements, we suppose that material points on either side of the interface are allowed to separate and move away from each other in the normal direction, the displacement being proportional to the traction normal to the interface. When material points on either side of the interface are in contact, the normal component of the traction is directed into the fiber. We apply these nonlinear interface conditions and solve for the displacement inside the shaft to find that the displacement and stress are identical to the displacement and stress obtained using the previous linear interface model. This shows that imperfect adhesion normal to the fiber-matrix interface does not affect the torsional rigidity of the shaft. These results are presented in Section 8.

The paper is organized as follows. In Section 2 we obtain the form of solution for the torsion boundary-value problem. The solution is given in terms of a discontinuous warping function. Here we introduce the stress potential and formulate the torsional rigidity in terms of it. In Section 3 we derive inequalities for the torsional rigidity that are the essential ingredients in the proofs of Theorems 1.1 through 1.7. The upper inequality for the rigidity is given in terms of the *ratio of surface traction to bulk stress*. In Section 4 we find the stress potential for the circular shaft reinforced with a single concentric circular fiber. We obtain an analytic expression for the stress potential for a shaft with circular cross section of radius $R$ reinforced with circular fibers with common radius equal to $\mathrm{R}_{\mathrm{cr}}$. In Section 5 we prove Theorems 1.1, 1.3. In Section 6 we prove the optimal design theorem, the isoperimetric inequality, and two of its consequences given by Theorems 1.4 and 1.5. The extreme cases $\alpha = 0$ and $\alpha = \infty$ are addressed in Section 7. In Section 8 we examine imperfect interfaces for which there are discontinuities in both the normal and tangential components of the displacement across the interface. We give a sketch of the proof for the existence and uniqueness of a solution for the torsion boundary-value problem in the Appendix.

## 2. The Torsion Boundary-Value Problem and the Torsional Rigidity

In this Section we formulate the interface conditions and the equations of mechanical equilibrium for the elastic displacement. It is shown that the displacement along the axis of the shaft is characterized by a discontinuous warping function. We consider a shaft reinforced with $N$ fibers each with cross section denoted by $\Sigma_i$, $i = 1, \ldots, N$. The union of all fiber-matrix interfaces is written as $\Gamma$. The jump in a quantity $q$ across $\Gamma$ is denoted by $[q] = q_{\mathrm{f}} - q_{\mathrm{m}}$, where $q_{\mathrm{f}}$ is the trace of the quantity on the fiber side and $q_{\mathrm{m}}$ is the trace on the matrix side. On the interface, the elastic deformation is decomposed into normal and tangential components given by $u_n = \mathbf{u} \cdot \boldsymbol{n}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n}$, where $\boldsymbol{n}$ is the unit normal pointing out of the fiber domain into the matrix. The stress tensor inside the composite shaft is denoted by $\sigma_{ij}$ and on the interface the traction is decomposed into normal and tangential components given by $\sigma_n = \sigma_{ij} n_i n_j$ and $(\sigma_\tau)_i = \sigma_{ij} n_j - (\sigma_{kl} n_k n_l) n_i$. Inside each phase we have the equilibrium condition

$$\partial_j \sigma_{ij} = 0, \tag{2.1}$$

and on the fiber-matrix interface we have the imperfect bonding conditions described by

$$[u_n] = 0 \quad \text{on } \Gamma, \tag{2.2}$$

$$[\sigma_{ij} n_j] = 0 \quad \text{on } \Gamma, \tag{2.3}$$

$$\sigma_\tau = -\alpha [\boldsymbol{u}_\tau] \quad \text{on } \Gamma. \tag{2.4}$$

The constitutive law is given by $\sigma = \mathscr{C} e(\boldsymbol{u})$, where $e(\boldsymbol{u})$ is the strain matrix given by $e(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^t)$ and $\mathscr{C}$ is the isotropic elasticity tensor taking different values in each phase. The elasticity tensor is specified by bulk and shear moduli $\kappa_{\mathrm{f}}$ and $\mathrm{G}_{\mathrm{f}}$ inside the fibers and in the matrix by $\kappa_{\mathrm{m}}$ and $\mathrm{G}_{\mathrm{m}}$. The equilibrium condition (2.1) together with the constitutive law, interface conditions (2.2)–(2.4), and boundary conditions given in Section 1 constitute a well-posed boundary-value problem for the elastic displacement. The solution is easily seen to be unique up to a constant translation parallel to the axis of the shaft. Existence of a solution to this problem follows easily from its variational formulation. A short outline of the existence proof is given in the Appendix.

We may proceed as in the perfectly bonded case to find that the solution is of Saint-Venant type. That is, the displacement in the shaft is given by

$$u_1 = -\theta x_3 x_2, \quad u_2 = \theta x_3 x_1, \tag{2.5}$$

$$u_3 = \theta w(x_1, x_2). \tag{2.6}$$

The function $w(x_1, x_2)$ is analogous to the warping function appearing in the torsion problem with perfectly bonded interfaces. However, unlike the perfectly bonded case the warping function introduced here can be discontinuous across the fiber-matrix interface. To make this precise we consider the union of all fiber cross sections $\cup_{i=1}^{N} \Sigma_i$ and denote the boundary of the union of all fiber cross sections by $J$. We allow the warping function to have jump discontinuities across $J$. The set of all points in $\Omega$ not on $J$ is denoted by $\Omega \setminus J$. The warping function is assumed to belong to the space of square integrable functions with square-integrable first derivitaves on the region $\Omega \setminus J$. This space is denoted by $H^1(\Omega \setminus J)$.

Equations (2.5) and (2.6) imply that the only nonzero components of the strain tensor are given by

$$e_{13} = \tfrac{1}{2}\theta(\partial_{x_1} w - x_2), \quad e_{23} = \tfrac{1}{2}\theta(\partial_{x_2} w + x_1) \quad \text{in } \Omega \setminus J. \tag{2.7}$$

The nonzero components of the stress tensor are

$$\sigma_{13} = \theta \mathrm{G}(x)(\partial_{x_1} w - x_2), \quad \sigma_{23} = \theta \mathrm{G}(x)(\partial_{x_2} w + x_1) \quad \text{in } \Omega \setminus J. \tag{2.8}$$

Here $\mathrm{G}(x)$ is the piecewise constant shear modulus taking the values $\mathrm{G_m}$ and $\mathrm{G_f}$ in the matrix and fiber, respectively. Substitution of (2.5) and (2.6) into the interface conditions (2.2) and (2.3) gives

$$\sigma_n = 0 \quad \text{on } J, \tag{2.9}$$

$$[\mathrm{G}(x)(\nabla w + \tilde{v}) \cdot n] = 0 \quad \text{on } J, \tag{2.10}$$

$$\mathrm{G_f}(\nabla w + \tilde{v})_f \cdot n = -\alpha[w] \quad \text{on } J. \tag{2.11}$$

Here $\nabla w = (\partial_{x_1} w, \partial_{x_2} w)^{\mathrm{t}}$, $\tilde{v} = (-x_2, x_1)^{\mathrm{t}}$, and $n = (n_1, n_2)^{\mathrm{t}}$ is the unit normal pointing from the fiber into the matrix. The traction-free condition on the sides of the shaft gives

$$\partial_n w = -n \cdot \tilde{v} \quad \text{on } \partial\Omega, \tag{2.12}$$

and the equilibrium condition $\partial_j \sigma_{ij} = 0$ gives

$$\Delta w = 0 \text{ in } \Omega \setminus J. \tag{2.13}$$

Equations (2.10)–(2.13) determine the warping function up to an additive constant.

We introduce the harmonic function $\phi$ conjugate to the warping function $w$ on the region $\Omega \setminus J$. This function is defined uniquely up to an additive constant inside each fiber and in the matrix. The stress potential $\Phi$ is defined as

$$\Phi = \mathrm{G}(x)\big(\phi - \tfrac{1}{2}(x_1^2 + x_2^2)\big). \tag{2.14}$$

We easily calculate for all points in $\Omega \setminus J$ that

$$\nabla\Phi = -R\mathrm{G}(x)(\nabla w + \tilde{v}) \tag{2.15}$$

where $R$ is the rotation matrix associated with a clockwise rotation of $\frac{\pi}{2}$ radians. Relations (2.14) and (2.15) allow us to recover the boundary-value problem for the stress potential from that of the warping function. From (2.14) we obtain

$$G^{-1}(x)\Delta\Phi = -2 \quad \text{in } \Omega \setminus J. \tag{2.16}$$

Application of (2.10) and (2.15) gives

$$0 = [G(x)(\nabla w + \tilde{v}) \cdot n] = [R\nabla\Phi \cdot n] \quad \text{on } J \tag{2.17}$$

and we find that

$$[\partial_\tau \Phi] = 0 \quad \text{on } J. \tag{2.18}$$

Here $\partial_\tau$ indicates tangential differentiation along the interface. It follows from (2.18) that adjustment by a constant in each fiber (if necessary) gives $[\Phi] = 0$ across the fiber-matrix interface. Thus the gradient of $\Phi$ is square-integrable over the whole domain $\Omega$, and $\Phi$ lies in the Sobolev space $H^1(\Omega)$. From (2.12) we find that $\partial_\tau\Phi = 0$ on $\partial\Omega$, and so $\Phi$ is a constant on $\partial\Omega$. We fix the last constant at our disposal to set $\Phi = 0$ on $\partial\Omega$. Lastly, we recover the transmission conditions satisfied by the derivatives of the stress potential across the interface. We return to equation (2.15) and apply standard trace theorems to find that

$$[G^{-1}\partial_n\Phi] = -[R(\nabla w + \tilde{v}) \cdot n] \quad \text{on } J. \tag{2.19}$$

Noting that $R^{-1} = -R$ and $Rn = \tau$ where $\tau$ is the unit tangent to $J$, we have

$$[G^{-1}\partial_n\Phi] = [\partial_\tau w] \quad \text{on } J. \tag{2.20}$$

On the other hand from, (2.11) we have

$$G_f(\nabla w + \tilde{v})_f \cdot n = -\alpha[w] \quad \text{on } J \tag{2.21}$$

and, since $R\nabla\Phi_f \cdot n = G_f(\nabla w + \tilde{v})_f \cdot n$ on $J$, we obtain

$$-\partial_\tau\Phi = -\alpha(w_f - w_m) \quad \text{on } J. \tag{2.22}$$

(Here we recall from (2.18) that the tangential derivative of $\Phi$ is continuous across $J$.) When $J$ is sufficiently regular, we may differentiate (2.22) and apply (2.20) to find the desired transmission condition:

$$\alpha^{-1}\partial_\tau^2\Phi = [G^{-1}\partial_n\Phi]. \tag{2.23}$$

Collecting our results we find that the transmission condition $[\Phi] = 0$ on $J$, (2.16) and (2.23) together with the boundary condition $\Phi = 0$ on $\partial\Omega$ provide a well-posed boundary-value problem for the stress potential. Existence and uniqueness follows from an application of the Lax-Milgram Lemma; this is established in the work of PHAM HUY & SANCHEZ-PALENCIA [10].

The torsional rigidity can be expressed in terms of the stress potential. Substitution of the stress potential into equation (1.1) gives

$$T(A, \Omega, \alpha) = \int_\Omega G^{-1}(x)|\nabla\Phi|^2 dx + \alpha^{-1}\int_J |\partial_\tau\Phi|^2 dl. \tag{2.24}$$

To proceed with the analysis we formulate the torsional rigidity in terms of the following variational principle. We write

$$\mathrm{T}(A,\Omega,\alpha) = -2E(A,\Omega,\alpha) \tag{2.25}$$

where

$$E(A,\Omega,\alpha) = \min_{\varphi\in H_0^1(\Omega)} \left\{ \frac{1}{2}\left( \int_\Omega \mathrm{G}^{-1}(x)|\nabla\varphi|^2 dx + \alpha^{-1}\int_J |\partial_\tau\varphi|^2 dl \right) - 2\int_\Omega \varphi\, dx \right\}. \tag{2.26}$$

Here the minimizer is precisely the stress potential. For future reference we write a second variational principle for the torsional rigidity given by

$$\mathrm{T}(A,\Omega,\alpha) = \min_{w\in H^1(\Omega\backslash J)} \int_\Omega \mathrm{G}(x)|\nabla w + \tilde{v}|^2 dx + \alpha\int_J ([w])^2 dl. \tag{2.27}$$

Here the minimizer is precisely the warping function in the shaft.

## 3. Inequalities for the Torsional Rigidity

In this Section we introduce the tools used in proving Theorems 1.1 through 1.7. We fix the cross section $\Omega$ of the shaft and investigate the effects of adding a reinforcement fiber to an already existing fiber configuration. We denote the cross section of the existing fiber configuration by $A$ and the cross section of the fiber to be added by $\Sigma$. We suppose that the additional fiber is placed so as not to come into contact with other fibers and that its boundary does not touch the boundary of the shaft. The torsional rigidity of the original configuration is denoted by $\mathrm{T}(A,\Omega,\alpha)$. The rigidity associated with the added fiber is written as $\mathrm{T}(A\cup\Sigma,\Omega,\alpha)$. Next we introduce the torsional rigidity $S(\Sigma)$ of the fiber cross section $\Sigma$ filled with elastic material of unit shear stiffness. Here $S(\Sigma)$ is given in terms of the stress potential $\Psi$ by

$$S(\Sigma) = \int_\Sigma |\nabla\Psi|^2 dx, \tag{3.1}$$

where the stress potential satisfies

$$\Delta\Psi = -2 \quad \text{in } \Sigma, \tag{3.2}$$

and $\Psi = 0$ on $\partial\Sigma$. We denote the *ratio of surface traction to bulk stress* associated with the fiber cross section $\Sigma$ by $\beta_1(\Sigma)$ and state

**Theorem 3.1** (Upper rigidity inequality).. *If*

$$\beta_1(\Sigma) \geqq \mathrm{R}_{\mathrm{cr}}^{-1}, \tag{3.3}$$

*then*

$$\mathrm{T}(A\cup\Sigma,\Omega,\alpha) \leqq \mathrm{T}(A,\Omega,\alpha) + (\mathrm{G_f} - \mathrm{G_m})S(\Sigma). \tag{3.4}$$

Theorem 3.1 is established with the aid of the variational principle given by (2.26). We remark that the methods used to establish this inequality obtain when the fiber cross section is multiply connected. The idea of the proof is to estimate the quantity $\mathrm{E}(A, \Omega, \alpha)$ in terms of the energy $\mathrm{E}(A \cup \Sigma, \Omega, \alpha)$ associated with the additional fiber. We let $\mathscr{G}(x)$ denote the piecewise constant shear modulus for the configuration $A \cup \Sigma$. We regroup terms in the variational principle (2.26) and write

$$\begin{aligned}
&\mathrm{E}(A, \Omega, \alpha)\\
&= \min_{\varphi \in H_0^1(\Omega)} \left\{ \frac{1}{2}\left( \int_\Omega \mathscr{G}^{-1}(x)|\nabla\varphi|^2 dx + \alpha^{-1} \int_{\partial A \cup \partial\Sigma} |\partial_\tau \varphi|^2 dl \right) - 2\int_\Omega \varphi dx \right.\\
&\qquad \left. + \frac{1}{2}\left( \int_\Sigma (\mathrm{G_m}^{-1} - \mathrm{G_f}^{-1})|\nabla\varphi|^2 dx - \alpha^{-1}\int_{\partial\Sigma} |\partial_\tau\varphi|^2 dl \right) \right\}. \qquad (3.5)
\end{aligned}$$

We obtain an estimate for $\mathrm{E}(A, \Omega, \alpha)$ by substitution of a suitable trial field into (3.5). We introduce the stress potential $\tilde{\Phi}$ for the configuration $A \cup \Sigma$. The trial field $\varphi$ is chosen to match $\tilde{\Phi}$ outside the fiber cross section $\Sigma$ but inside we require that $\Delta\varphi = -2\mathrm{G_m}$. In other words, we choose $\varphi = \tilde{\Phi} + \delta$ where $\delta$ satisfies

$$\Delta\delta = -2(\mathrm{G_m} - \mathrm{G_f}) \quad \text{in } \Sigma, \qquad (3.6)$$

$\delta = 0$ on $\partial\Sigma$, and $\delta = 0$ outside of the fiber cross section. This choice of $\delta$ ensures that $\varphi$ is an admissible trial for the variational principle. We observe that $\partial_\tau\varphi = \partial_\tau\tilde{\Phi}$ on $\partial\Sigma$ and that substitution of $\varphi$ into (3.5) gives

$$\begin{aligned}
\mathrm{E}(A, \Omega, \alpha) &\leqq \frac{1}{2}\biggl( \int_{\Omega/\Sigma} \mathscr{G}^{-1}(x)|\nabla\tilde{\Phi}|^2 dx + \int_\Sigma \mathscr{G}^{-1}(x)|\nabla\tilde{\Phi} + \nabla\delta|^2 dx\\
&\quad + \alpha^{-1}\int_{\partial A \cup \partial\Sigma} |\partial_\tau\tilde{\Phi}|^2 dl \biggr) - 2\int_\Omega \tilde{\Phi} dx - 2\int_\Sigma \delta\, dx \qquad (3.7)\\
&\quad + \frac{1}{2}\biggl( \int_\Sigma (\mathrm{G_m}^{-1} - \mathrm{G_f}^{-1})|\nabla\varphi|^2 dx - \alpha^{-1}\int_{\partial\Sigma} |\partial_\tau\tilde{\Phi}|^2 dl \biggr).
\end{aligned}$$

We expand the second term on the right-hand side of (3.7) to find

$$\int_\Sigma \mathscr{G}^{-1}(x)|\nabla\tilde{\Phi} + \nabla\delta|^2 dx = \mathrm{G_f}^{-1}\int_\Sigma |\nabla\tilde{\Phi}|^2 dx + \mathrm{G_f}^{-1}\int_\Sigma |\nabla\delta|^2 dx + 4\int_\Sigma \delta\, dx. \qquad (3.8)$$

Substitution of (3.8) into (3.7) yields

$$\begin{aligned}
\mathrm{E}(A, \Omega, \alpha) &\leqq \mathrm{E}(A \cup \Sigma, \Omega, \alpha) + \frac{1}{2}\mathrm{G_f}^{-1}\int_\Sigma |\nabla\delta|^2 dx\\
&\quad + \frac{1}{2}\biggl( \int_\Sigma (\mathrm{G_m}^{-1} - \mathrm{G_f}^{-1})|\nabla\varphi|^2 dx - \alpha^{-1}\int_{\partial\Sigma} |\partial_\tau\tilde{\Phi}|^2 dl \biggr).
\end{aligned} \qquad (3.9)$$

Next we simplify the third term on the right-hand side of (3.9). We decompose the trial $\varphi$ into two parts: $\varphi = r + q$, where the function $r$ satisfies

$$\Delta r = 0 \quad \text{in } \Sigma, \qquad r = \tilde{\Phi} \quad \text{on } \partial\Sigma, \tag{3.10}$$

and $q$ satisfies

$$\Delta q = -2\mathrm{G_m} \quad \text{in } \Sigma, \qquad q = 0 \quad \text{on } \partial\Sigma. \tag{3.11}$$

We observe from the linearity of the torsion problems (3.2), (3.6), and (3.11) that $\delta = (\mathrm{G_m} - \mathrm{G_f})\Psi$ and $q = \mathrm{G_m}\Psi$. Noting that $\partial_\tau r = \partial_\tau \tilde{\Phi}$ on the fiber surface and substitution into (3.9) gives

$$\begin{aligned} \mathrm{E}(A, \Omega, \alpha) \leqq\ & \mathrm{E}(A \cup \Sigma, \Omega, \alpha) + \frac{1}{2}(\mathrm{G_f} - \mathrm{G_m})S(\Sigma) \\ & + \frac{1}{2}\left( \int_\Sigma ({\mathrm{G_m}}^{-1} - {\mathrm{G_f}}^{-1})|\nabla r|^2 dx - \alpha^{-1} \int_{\partial\Sigma} |\partial_\tau r|^2 dl \right). \end{aligned} \tag{3.12}$$

Since the function $r$ is harmonic in $\Sigma$, it is evident from (1.11) that

$$\int_\Sigma ({\mathrm{G_m}}^{-1} - {\mathrm{G_f}}^{-1})|\nabla r|^2 dx - \alpha^{-1} \int_{\partial\Sigma} |\partial_\tau r|^2 dl \leqq 0 \tag{3.13}$$

provided that $\beta_1(\Sigma) \geqq \mathrm{R_{cr}^{-1}}$, and the theorem follows.

In order to give the second inequality we introduce the stress potential $\Phi_A$ for a fiber configuration with cross section $A$. As before, we let $\Sigma$ represent the cross section of the fiber to be added. We note that the methods used here place no constraint on the connectivity of the cross section $\Sigma$. In fact, $\Sigma$ can be the cross section for a union of fibers with multiply connected cross sections. We state

**Theorem 3.2** (Lower rigidity inequality).

$$\begin{aligned} \mathrm{T}(A \cup \Sigma, \Omega, \alpha) \geqq\ & \mathrm{T}(A, \Omega, \alpha) + (\mathrm{G_f} - \mathrm{G_m})S(\Sigma) \\ & - \left(\alpha^{-1} \int_{\partial\Sigma} |\partial_\tau u|^2 dl - ({\mathrm{G_m}}^{-1} - {\mathrm{G_f}}^{-1}) \int_\Sigma |\nabla u|^2 dx\right), \end{aligned} \tag{3.14}$$

*where $u$ is harmonic inside $\Sigma$ and $u = \Phi_A$ on $\partial\Sigma$.*

To prove the theorem, we let $\tilde{\mathrm{G}}(x)$ denote the piecewise constant shear moduli inside the shaft associated with the configuration of fibers with cross section $A$. Rearranging terms gives

$$\begin{aligned} & \mathrm{E}(A \cup \Sigma, \Omega, \alpha) \\ & = \min_{\varphi \in H_0^1(\Omega)} \left\{ \frac{1}{2}\left( \int_\Omega \tilde{\mathrm{G}}(x)^{-1} |\nabla\varphi|^2 dx + \alpha^{-1} \int_{\partial A} |\partial_\tau \varphi|^2 dl \right) - 2\int_\Omega dx \right. \\ & \left. + \left( \alpha^{-1} \int_{\partial\Sigma} |\partial_\tau\varphi|^2 dl - ({\mathrm{G_m}}^{-1} - {\mathrm{G_f}}^{-1}) \int_\Sigma |\nabla\varphi|^2 dx \right) \right\}. \end{aligned} \tag{3.15}$$

We make the choice of trial function $\varphi \in H_0^1(\Omega)$ given by $\varphi = \Phi_A + v$ where

$$\Delta v = -2(\mathrm{G_f} - \mathrm{G_m}) \quad \text{in } \Sigma, \quad v = 0 \quad \text{on } \partial\Sigma, \quad v = 0 \quad \text{on } \Omega \setminus \Sigma. \tag{3.16}$$

For this choice we have $\Delta\varphi = -2\mathrm{G_f}$ in $\Sigma$ and $\varphi = \Phi_A$ on $\partial\Sigma$. Substitution of $\varphi$ into (3.15) gives

$$\begin{aligned} &\mathrm{E}(A \cup \Sigma, \Omega, \alpha) \\ &\quad \leqq \mathrm{E}(A, \Omega, \alpha) + \frac{1}{2}\mathrm{G_m}^{-1}\int_\Sigma |\nabla v|^2 dx \\ &\qquad + \frac{1}{2}\left(\alpha^{-1}\int_{\partial\Sigma} |\partial_\tau \Phi_A|^2 dl - (\mathrm{G_m}^{-1} - \mathrm{G_f}^{-1})\int_\Sigma |\nabla\varphi|^2 dx\right). \end{aligned} \tag{3.17}$$

We expand the last term in (3.17) by decomposing $\varphi$ into two parts $\varphi = w + u$, where

$$\Delta u = 0 \quad \text{in } \Sigma, \quad u = \Phi_A \quad \text{on } \partial\Sigma, \tag{3.18}$$

$$\Delta w = -2\mathrm{G_f} \quad \text{in } \Sigma, \quad w = 0 \quad \text{on } \partial\Sigma. \tag{3.19}$$

Substitution into (3.17) gives:

$$\begin{aligned} &\mathrm{E}(A \cup \Sigma, \Omega, \alpha) \\ &\leqq \mathrm{E}(A, \Omega, \alpha) + \frac{1}{2}\left(\mathrm{G_m}^{-1}\int_\Sigma |\nabla v|^2 dx + -(\mathrm{G_m}^{-1} - \mathrm{G_f}^{-1})\int_\Sigma |\nabla w|^2\right) \\ &\quad + \frac{1}{2}\left(\alpha^{-1}\int_{\partial\Sigma} |\partial_\tau u|^2 dl - (\mathrm{G_m}^{-1} - \mathrm{G_f}^{-1})\int_\Sigma |\nabla u|^2 dx\right). \end{aligned} \tag{3.20}$$

Next we observe that $v = (\mathrm{G_f} - \mathrm{G_m})\Psi$ and $w = \mathrm{G_f}\Psi$. Substitution into (3.20) gives

$$\begin{aligned} &\mathrm{E}(A \cup \Sigma, \Omega, \alpha) \\ &\quad \leqq \mathrm{E}(A, \Omega, \alpha) - \tfrac{1}{2}(\mathrm{G_f} - \mathrm{G_m})S(\Sigma) \\ &\qquad + \frac{1}{2}\left(\alpha^{-1}\int_{\partial\Sigma} |\partial_\tau u|^2 dl - (\mathrm{G_m}^{-1} - \mathrm{G_f}^{-1})\int_\Sigma |\nabla u|^2 dx\right), \end{aligned} \tag{3.21}$$

and the theorem follows.

In order to carry out the proof of Theorem 1.5 we shall need estimates for the ratio of surface traction to bulk stress of a fiber cross section. For the purposes of this analysis, several useful estimates and isoperimetric inequalities already exist for the second Stekloff eigenvalue for two-dimensional domains; see [9]. For planar domains we show that the second Stekloff eigenvalue agrees with the ratio of surface traction to bulk stress. We denote the second Stekloff eigenvalue associated with a cross section $\Sigma$ by $\rho_2$. This number is the first nonzero eigenvalue for the problem [14]:

$$\Delta f = 0 \quad \text{in } \Sigma, \quad \partial_n f = \rho f \quad \text{on } \partial\Sigma. \tag{3.22}$$

**Theorem 3.3.** *For any two-dimensional domain* $\Sigma$ *with Lipschitz boundary,*

$$\beta_1 = \rho_2. \tag{3.23}$$

To establish the theorem we introduce the Rayleigh quotient

$$\rho_2 = \min_{v \in K} \frac{\int_{\partial\Sigma} |\partial_n v|^2 dl}{\int_\Sigma |\nabla v|^2 dx}, \tag{3.24}$$

where

$$K = \left\{ v | \, \Delta v = 0 \quad \text{in } \Sigma, \; \int_{\partial\Sigma} v \, dl = 0 \right\}. \tag{3.25}$$

We denote the minimizer of the Rayleigh quotient (1.10) by $\widehat{\varphi}$. The unit normal $n$ and unit tangent $\tau$ on the boundary are related by $\tau = Rn$, where $R$ is the matrix associated with a clockwise rotation of $\frac{\pi}{2}$ radians. The harmonic function conjugate to $\widehat{\varphi}$ is denoted by $\widehat{u}$, and

$$\beta_1 = \frac{\int_{\partial\Sigma} |\partial_\tau \widehat{\varphi}|^2 dl}{\int_\Sigma |\nabla \widehat{\varphi}|^2 dx} = \frac{\int_{\partial\Sigma} |\partial_n \widehat{u}|^2 dl}{\int_\Sigma |\nabla \widehat{u}|^2 dx}, \tag{3.26}$$

and so $\beta_1 \geqq \rho_2$. On the other hand, we may start with the Rayleigh quotient for $\rho_2$ and argue as above to find $\rho_2 \geqq \beta_1$ and the theorem follows.

It should be noted that Theorem 3.3 does not hold for domains in three dimensions. Indeed for a sphere of radius $a$, separation of variables gives $\beta_1 = 2/a$ and $\rho_2 = 1/a$. More generally for domains with sufficiently regular boundaries we have the estimate $\beta_1 \leqq \nu_1/\rho_2$, where $\nu_1$ is the first nonzero eigenvalue of the Laplace-Beltrami operator on the boundary of the domain. Equality is seen to hold for any sphere.

## 4. Torsional Rigidity for Special Fiber Configurations

We consider a shaft with circular cross section of radius $R$ reinforced with a single fiber with circular cross section of radius $a$. The fiber is centered inside the shaft, i.e., the cross sections of the fiber and shaft are concentric circles. Calculation shows that the stress potential for this configuration is given by

$$\psi = \begin{cases} f(x), & x \text{ outside the fiber}, \\ f(x) + v(x), & x \text{ in the fiber}, \end{cases} \tag{4.1}$$

where

$$f(x) = -\tfrac{1}{2}\mathrm{G_m}|x|^2 + \tfrac{1}{2}\mathrm{G_m}R^2, \tag{4.2}$$

$$v(x) = -\tfrac{1}{2}(\mathrm{G_f} - \mathrm{G_m})|x|^2 + \tfrac{1}{2}(\mathrm{G_f} - \mathrm{G_m})a^2, \tag{4.3}$$

where $|x|^2 = x_1^2 + x_2^2$. The associated warping function for this configuration is

$$w(x_1, x_2) = \text{constant}. \tag{4.4}$$

It is evident that the traction vanishes at the fiber-matrix interface and that the displacement is continuous everywhere in the shaft. Furthermore, we see that the stress potential and displacements are independent of the interfacial tangential stiffness $\alpha$. A straightforward calculation shows that the torsional rigidity for this configuration is given by (1.2).

Next we establish Theorem 1.2. The theorem follows from the calculation of the potential for a system of $N$ fibers with common radius $\mathrm{R_{cr}}$. To start, we consider $N$ fibers with common radius $a$ and fiber centers $\hat{x}_i$, $i = 1, \ldots, N$ and look for a solution of the form,

$$\psi = \begin{cases} f(x), & x \text{ outside the fiber,} \\ f(x) + v_i(x), & x \text{ inside the } i^{\text{th}} \text{ fiber} \end{cases} \tag{4.5}$$

where $f(x)$ is given by (4.2) and where

$$v_i(x) = -\tfrac{1}{2}(\mathrm{G_f} - \mathrm{G_m})|x - \hat{x}_i|^2 + \tfrac{1}{2}(\mathrm{G_f} - \mathrm{G_m})a^2, \qquad i = 1, \ldots, N. \tag{4.6}$$

The potential $\psi$ is in $H_0^1(\Omega)$ and satisfies the equilibrium conditions: ${\mathrm{G_m}}^{-1}\Delta\psi = -2$ in the matrix and ${\mathrm{G_f}}^{-1}\Delta\psi = -2$ in the fibers. The final condition to be satisfied is the transmission condition (2.23). Calculation shows that on the $i^{\text{th}}$ fiber,

$$\begin{aligned} &\alpha^{-1}\partial_\tau^2\psi - [G^{-1}\partial_n\psi] \\ &\quad = a^{-1}\mathrm{G_m}\left(\frac{\alpha^{-1}}{a}\hat{x}_i \cdot (x - \hat{x}_i) + ({\mathrm{G_f}}^{-1} - {\mathrm{G_m}}^{-1})x \cdot (x - \hat{x}_i)\right) \\ &\qquad + a^{-1}(\mathrm{G_f} - \mathrm{G_m})|x - \hat{x}_i|^2. \end{aligned} \tag{4.7}$$

We find that the right-hand side vanishes for the choice $a = \mathrm{R_{cr}}$, and so (2.23) is satisfied for this value of fiber radius. We conclude that the stress potential is given by

$$\psi = \begin{cases} -\tfrac{1}{2}\mathrm{G_m}|x|^2 + \tfrac{1}{2}\mathrm{G_m}R^2, & x \text{ outside the fibers,} \\ f(x) - \tfrac{1}{2}(\mathrm{G_f} - \mathrm{G_m})|x - \hat{x}_i|^2 + \tfrac{1}{2}(\mathrm{G_f} - \mathrm{G_m})\mathrm{R_{cr}^2}, & x \text{ inside the } i^{\text{th}} \text{ fiber.} \end{cases} \tag{4.8}$$

Substitution of (4.8) into (2.24) gives the torsional rigidity (1.7), and Theorem 1.2 follows.

## 5. Fiber Size and Optimal Configurations for Maximum Torsional Rigidity

In this section we focus on shafts of circular cross section reinforced with fibers of circular cross section. The cross-sectional radius of each fiber may be different and the radius of the $i^{\text{th}}$ fiber is denoted by $a_i$. For $N$ fibers with centers $\hat{x}_i$ the region occupied by fibers is written, $\cup_{i=1}^N D_{\hat{x}_i}(a_i)$, where $D_{\hat{x}_i}(a_i)$ is a disk of radius

$a_i$ centered at $\hat{x}_i$ and represents the cross section of the $i^{\text{th}}$ fiber. The torsional rigidity associated with this configuration is $\mathrm{T}(\cup_{i=1}^N D_{\hat{x}_i}(a_i), \Omega, \alpha)$. The following theorem gives a lower bound on the torsional rigidity that depends upon the configuration of the fibers.

**Theorem 5.1** (Lower bound on the torsional rigidity).

$$\begin{aligned} &\mathrm{T}(\cup_{i=1}^N D_{\hat{x}_i}(a_i), \Omega, \alpha) \\ &\quad \geqq \tfrac{\pi}{2}\mathrm{G_m} R^4 + (\mathrm{G_f} - \mathrm{G_m}) \sum_{i=1}^N \tfrac{\pi}{2} a_i^4 \\ &\quad - \left( (\mathrm{G_m}^{-1} - \mathrm{G_f}^{-1}) \sum_{i=1}^N \pi \mathrm{G_m}^2 a_i |\hat{x}_i|^2 (\mathrm{R_{cr}} - a_i) \right). \end{aligned} \tag{5.1}$$

*Remark.* It is interesting to note that if all fibers have the same cross-sectional radius $a$ and if $a \geqq \mathrm{R_{cr}}$, then the lower bound is largest for configurations with sphere centers placed as far away from the center of the shaft as possible.

To prove this theorem we apply the energy dissipation inequality given in Theorem 3.2 with the convention that the "cross section" to be added is the union of all fiber cross sections $\cup_{i=1}^N D_{\hat{x}_i}(a_i)$ and the original configuration is the unreinforced shaft. The stress potential inside the unreinforced shaft made of homogeneous isotropic elastic material with shear stiffness $\mathrm{G_m}$ is $-\frac{1}{2}\mathrm{G_m}|x|^2 + \frac{1}{2}\mathrm{G_m}R^2$. Its torsional rigidity is $\frac{\pi}{2}\mathrm{G_m}R^4$. Appealing to Theorem 3.2 gives

$$\begin{aligned} &\mathrm{T}(\cup_{i=1}^N D_{\hat{x}_i}(a_i), \Omega, \alpha) \\ &\quad \geqq \tfrac{\pi}{2}\mathrm{G_m} R^4 + (\mathrm{G_f} - \mathrm{G_m}) S(\cup_{i=1}^N D_{x_i}(a_i)) \\ &\quad - \left( \alpha^{-1} \int_{\partial\Sigma} |\partial_\tau u|^2 dl - (\mathrm{G_m}^{-1} - \mathrm{G_f}^{-1}) \int_\Sigma |\nabla u|^2 dx \right). \end{aligned} \tag{5.2}$$

Here $u$ is harmonic inside the union $\cup_{i=1}^N D_{\hat{x}_i}(a_i)$ and $u = -\frac{1}{2}\mathrm{G_m}|x|^2 + \frac{1}{2}\mathrm{G_m}R^2$ on the matrix-fiber boundary. It is easily seen that inside each fiber $u = -\frac{1}{2}\mathrm{G_m}(|x|^2 - |x - \hat{x}_i|^2) + \frac{1}{2}\mathrm{G_m}(R^2 - a_i^2)$. We have

$$S(\cup_{i=1}^N D_{\hat{x}_i}(a_i)) = \sum_{i=1}^N S(D_{\hat{x}_i}(a_i)) = \sum_{i=1}^N \tfrac{\pi}{2} a_i^4. \tag{5.3}$$

Finally, we substitute $u$ into (5.2) to obtain (5.1) and the theorem follows.

From Theorem 5.1 it is evident that we have

**Corollary 5.1.** *If the fiber centers and radii satisfy*

$$\sum_{i=1}^N a_i |\hat{x}_i|^2 (\mathrm{R_{cr}} - a_i) \leqq 0, \tag{5.4}$$

*then the torsional rigidity satisfies*

$$\mathrm{T}(\cup_{i=1}^{N} D_{\hat{x}_i}(a_i), \Omega, \alpha) \geqq \tfrac{\pi}{2}\mathrm{G_m} R^4 + (\mathrm{G_f} - \mathrm{G_m}) \sum_{i=1}^{N} \tfrac{\pi}{2} a_i^4, \tag{5.5}$$

*where* (5.5) *holds with strict inequality when* (5.4) *does.*

We consider a single fiber configuration and establish Theorem 1.1. The upper energy dissipation inequality (Theorem 3.1) is applied to establish the first part of Theorem 1.1. To use the theorem we note that the fiber to be added has a cross section given by a disk of radius $a$ with center $\hat{x}$ and that its surface to volume dissipation is $a^{-1}$. The original configuration is simply the unreinforced shaft filled with pure matrix material with shear stiffness $\mathrm{G_m}$. When $a^{-1} \geqq \mathrm{R_{cr}^{-1}}$, Theorem 3.1 implies that

$$\mathrm{T}(D_{\hat{x}}(a), D_0(R), \alpha) \leqq \tfrac{\pi}{2}\mathrm{G_m} R^4 + (\mathrm{G_f} - \mathrm{G_m}) S(D_{\hat{x}}(a)). \tag{5.6}$$

The first part of Theorem 1.1 follows by noting that $S(D_{\hat{x}}(a)) = \frac{\pi}{2}a^4$. The second part of Theorem 1.1 is a statement of the case $N = 1$ given in Corollary 5.1.

We now establish Theorem 1.3. It is evident that inequality (1.9) of Theorem 1.3 is an immediate corollary of Theorem 5.1. To establish (1.8) we apply Theorem 3.1 iteratively. Recalling that each fiber cross section is a disk of radius $a$ with center $\hat{x}_i$ we write the union of fiber cross sections as $\cup_{i=1}^{N} D_{\hat{x}_i}(a)$. In the context of Theorem 3.1 we set $A = \cup_{i=1}^{N-1} D_{\hat{x}_i}(a)$ and $\Sigma = D_{\hat{x}_N}(a)$. We suppose that $a \leqq \mathrm{R_{cr}}$ and apply Theorem 3.1 to obtain

$$\mathrm{T}(\cup_{i=1}^{N} D_{\hat{x}_i}(a), D_0(R), \alpha) \leqq \mathrm{T}(\cup_{i=1}^{N-1} D_{\hat{x}_i}(a), D_0(R), \alpha) + (\mathrm{G_f} - \mathrm{G_m}) S(D_{\hat{x}_N}(a)). \tag{5.7}$$

We then apply Theorem 3.1 to the term $\mathrm{T}(\cup_{i=1}^{N-1} D_{\hat{x}_i}(a), D_0(R), \alpha)$ to obtain a similar estimate. Proceeding iteratively we obtain

$$\mathrm{T}(\cup_{i=1}^{N} D_{\hat{x}_i}(a), D_0(R), \alpha) \leqq \mathrm{T}(\emptyset, D_0(R), \alpha) + (\mathrm{G_f} - \mathrm{G_m}) \sum_{i=1}^{N} S(D_{\hat{x}_i}(a)). \tag{5.8}$$

Here $\mathrm{T}(\emptyset, D_0(R), \alpha)$ is the torsional rigidity of the unreinforced shaft filled with isotropic material with shear stiffness $\mathrm{G_m}$. The theorem follows recalling that $\mathrm{T}(\emptyset, D_0(R), \alpha) = \frac{\pi}{2}\mathrm{G_m} R^4$ and $S(D_{\hat{x}_i}(a)) = \frac{\pi}{2}a^4$.

## 6. An Isoperimetric Inequality and an Upper Bound for the Torsional Rigidity

In this section we prove Theorems 1.6, 1.4, and 1.5. A minor modification of the proof of Theorem 1.6 gives Theorem 1.7. The proof of Theorem 1.6 proceeds in two steps. The first step is an iterative application of Theorem 3.1 to obtain an upper bound on the torsional rigidity. The upper bound is given in terms of the torsional rigidity of the unreinforced shaft and the torsional rigidity of each fiber. We apply the proposition of Saint Venant (established by POLYA [11]) to the torsional rigidity of each fiber to complete the proof.

We recall that each fiber cross section is denoted by $\Sigma_i$, where $i = 1, \ldots, N$. To apply Theorem 3.1 we set $A = \cup_{i=1}^{N-1} \Sigma_i$ and $\Sigma = \Sigma_N$. From our hypothesis we have $\beta_1(\Sigma_N) \geqq \mathrm{R}_{\mathrm{cr}}^{-1}$ and we apply Theorem 3.1 to find that

$$\mathrm{T}(\cup_{i=1}^{N} \Sigma_i, \Omega, \alpha) \leqq \mathrm{T}(\cup_{i=1}^{N-1} \Sigma_i, \Omega, \alpha) + (\mathrm{G_f} - \mathrm{G_m}) S(\Sigma_N). \tag{6.1}$$

Noting that $\beta(\Sigma_{N-1}) \geqq \mathrm{R}_{\mathrm{cr}}^{-1}$ we obtain a similar estimate for $\mathrm{T}(\cup_{i=1}^{N-1} \Sigma_i, \Omega, \alpha)$. We iterate these arguments to find

$$\mathrm{T}(\cup_{i=1}^{N} \Sigma_i, \Omega, \alpha) \leqq \mathrm{T}(\emptyset, \Omega, \alpha) + (\mathrm{G_f} - \mathrm{G_m}) \sum_{i=1}^{N} S(\Sigma_i). \tag{6.2}$$

Here $\mathrm{T}(\emptyset, \Omega, \alpha)$ is the torsional rigidity of the unreinforced shaft filled with material with shear stiffness $\mathrm{G_m}$ and $S(\Sigma_i)$ is the torsional rigidity of the $i^{\text{th}}$ fiber filled with material of unit shear stiffness.

We apply POLYA's result [11] to the quantities: $\mathrm{T}(\emptyset, \Omega, \alpha), S(\Sigma_1), \ldots, S(\Sigma_N)$ to obtain

$$\mathrm{T}(\emptyset, \Omega, \alpha) \leqq \pi \frac{\mathrm{G_m}}{2} R^4,$$

$$S(\Sigma_i) \leqq \pi \frac{a_i^4}{2}, \; i = 1, \ldots, N,$$

where the area of $\Omega$ is $\pi R^2$ and the area of $\Sigma_i$ is $\pi a_i^2$. Application of these inequalities to (6.2) gives

$$\mathrm{T}(\cup_{i=1}^{N} \Sigma_i, \Omega, \alpha) \leqq \tfrac{\pi}{2} \mathrm{G_m} R^4 + (\mathrm{G_f} - \mathrm{G_m}) \sum_{i=1}^{N} \tfrac{\pi}{2} a_i^4. \tag{6.3}$$

The joint cross-sectional area of the fibers is $\pi \sum_{i=1}^{N} a_i^2$. We consider a single circular fiber of radius $a$ having the same area as the joint cross-sectional area of the fibers, i.e., $\pi a^2 = \pi \sum_{i=1}^{N} a_i^2$. It follows immediately that $a^4 \geqq \sum_{i=1}^{N} a_i^4$ and we obtain

$$\mathrm{T}(\cup_{i=1}^{N} \Sigma_i, \Omega, \alpha) \leqq \tfrac{\pi}{2} \mathrm{G_m} R^4 + (\mathrm{G_f} - \mathrm{G_m}) \tfrac{\pi}{2} a^4. \tag{6.4}$$

The theorem follows noting that the right-hand side of (6.4) is precisely the torsional rigidity of a concentric circular fiber in a circular shaft.

Theorem 1.5 follows from Theorem 1.6 with the aid of an isoperimetric inequality on the second Stekloff eigenvalue obtained by PAYNE [9].

**Theorem 6.1** (PAYNE). *If $D$ is a convex domain with analytic boundary whose curvature is denoted by $\kappa$, then $\rho_2(D) \geqq \kappa_{\min}$. Equality holds only for the circle.*

Here $\kappa_{\min}$ is the minimum value of the curvature on the boundary. In Section 3 the ratio of surface traction to bulk stress was shown to be identical to the second Stekloff eigenvalue. We denote the maximum radius of curvature by $a_{\max}$. From PAYNE's isoperimetric inequality we find that $a_{\max} \geqq \beta_1^{-1}$, and Theorem 1.5 follows from Theorem 1.6. It is evident that Theorem 1.4 is an immediate corollary of Theorem 1.5. It is also easy to see that Theorem 1.4 follows directly from Theorem 1.6 by means of elementary arguments.

To prove Theorem 1.7 we return to inequality (6.3) and consider a single circular fiber of radius $\breve{a}$ such that $\breve{a}^4 = \Sigma_{i=1}^N a_i^4$. Substitution of $\breve{a}$ into (6.3) gives

$$\mathrm{T}(\cup_{i=1}^N \Sigma_i, \Omega, \alpha) \leqq \tfrac{\pi}{2} \mathrm{G_m} R^4 + (\mathrm{G_f} - \mathrm{G_m}) \tfrac{\pi}{2} \breve{a}^4, \tag{6.5}$$

and the theorem follows.

## 7. The Extreme Cases of No Tangential Interfacial Adhesion and Perfect Bonding

It is shown here that the extreme cases of complete loss of adhesion and perfect bonding are given by the limits $\alpha \to 0$ and $\alpha \to \infty$ respectively. With these stability results in hand, we argue for the existence of an isoperimetric inequality of the type given in Theorem 1.6 for the case $\alpha = 0$. We show that there is no such isoperimetric inequality for the case $\alpha = \infty$.

We consider first the extreme case of total loss of adhesion in the directions tangential to the fibers. This behavior corresponds to the loss of tangential stiffness, i.e., $\alpha = 0$. The torsional rigidity for this case is denoted by $\mathrm{T}(A, \Omega, 0)$. It is given by the variational formulation

$$\mathrm{T}(A, \Omega, 0) = \min_{w \in H^1(\Omega \setminus J)} \int_\Omega G(x) |\nabla w + \tilde{v}|^2 dx. \tag{7.1}$$

Here $A$ is the union of all fiber cross sections, i.e., $A = \cup_{i=1}^N \Sigma_i$. The minimizer corresponds to the warping function inside the shaft. Taking the first variation of (7.1) gives the boundary-value problem for the warping function $w(x_1, x_2)$:

$$\Delta w = 0 \quad \text{on } \Omega \setminus J, \tag{7.2}$$

$$\mathrm{G_m}(\nabla w + \tilde{v}) \cdot n = 0 \quad \text{on } \partial\Omega, \tag{7.3}$$

$$\mathrm{G_m}(\nabla w + \tilde{v})_\mathrm{m} \cdot n = \mathrm{G_f}(\nabla w + \tilde{v})_\mathrm{f} \cdot n = 0 \quad \text{on } J. \tag{7.4}$$

The associated stress potential $\Phi$ satisfies

$$\mathrm{G_m}^{-1} \Delta\Phi = -2, \ \text{outside the fibers}, \tag{7.5}$$

$$\mathrm{G_f}^{-1} \Delta\Phi = -2, \ \text{inside the fibers}, \tag{7.6}$$

$$\Phi = c_i \quad \text{on } \partial\Sigma_i, \quad \Phi = 0 \quad \text{on } \partial\Omega. \tag{7.7}$$

Here $c_i$, $i = 1, \ldots, N$, are constants and $\partial\Sigma_i$ denotes the boundary of the $i^{\text{th}}$ fiber cross section. The torsional rigidity is given in terms of the stress potential by

$$\mathrm{T}(A, \Omega, 0) = \int_{\Omega} \mathrm{G}^{-1}(x)|\nabla \Phi|^2 dx. \tag{7.8}$$

The limiting behavior as the interfacial shear stiffness tends to zero is expressed by the following stability result:

**Theorem 7.1.**

$$\lim_{\alpha \to 0} \mathrm{T}(A, \Omega, \alpha) = \mathrm{T}(A, \Omega, 0). \tag{7.9}$$

We give a proof. It is evident from its variational formulation that $\mathrm{T}(A, \Omega, \alpha)$ decreases monotonically as $\alpha$ tends to zero. From (2.27), (7.1), and monotonicity, we have that the limit of $\mathrm{T}(A, \Omega, \alpha)$ exists as $\alpha$ tends to zero and

$$\lim_{\alpha \to 0} \mathrm{T}(A, \Omega, \alpha) \geqq \mathrm{T}(A, \Omega, 0). \tag{7.10}$$

On the other hand, from (2.27) we have

$$\mathrm{T}(A, \Omega, \alpha) \leqq \left( \int_{\Omega} \mathrm{G}(x)|\nabla w + \tilde{v}|^2 dx + \alpha \int_J ([w])^2 dl \right), \tag{7.11}$$

for any $w$ in $H^1(\Omega \setminus J)$. Passing to the limit $\alpha = 0$ in (7.11) and subsequently minimizing over $w$ gives

$$\lim_{\alpha \to 0} \mathrm{T}(A, \Omega, \alpha) \leqq \mathrm{T}(A, \Omega, 0), \tag{7.12}$$

and the theorem is proved.

Next we investigate the asymptotic behavior of the torsional rigidity in the $\alpha = \infty$ limit. The torsional rigidity for a perfectly bonded composite is denoted by $\mathrm{T}(A, \Omega, \infty)$. It is given by the well-known variational formulation

$$\mathrm{T}(A, \Omega, \infty) = -2\mathrm{E}(A, \Omega, \infty) \tag{7.13}$$

where

$$\mathrm{E}(A, \Omega, \infty) = \min_{\psi \in H_0^1(\Omega)} \left\{ \frac{1}{2} \left( \int_{\Omega} \mathrm{G}^{-1}|\nabla \psi|^2 dx \right) - 2 \int_{\Omega} \psi dx \right\}. \tag{7.14}$$

The minimizer is the stress potential in the shaft.

The limiting behavior as $\alpha$ tends to $\infty$ is expressed by the following stability result:

**Theorem 7.2.**

$$\lim_{\alpha \to \infty} \mathrm{T}(A, \Omega, \alpha) = \mathrm{T}(A, \Omega, \infty). \tag{7.15}$$

The proof of this result follows the same lines as the proof given for Theorem 7.1. Here use is made of the variational formulation given by equation (2.26).

Collecting our results we have

$$\mathrm{T}(A, \Omega, \infty) \geqq \mathrm{T}(A, \Omega, \alpha) \geqq \mathrm{T}(A, \Omega, 0) \tag{7.16}$$

for $0 < \alpha < \infty$. This expresses the intuitive notion that shafts with perfectly bonded fibers are more rigid than ones with imperfectly bonded fibers and that the rigidity of a bar reinforced with partially adhesive fibers is greater than when there is no tangential adhesion at the interface. An extreme case corresponds to a shaft with circular cross section reinforced with a concentric circular fiber. For this configuration the stress potential is independent of the interfacial shear stiffness over the complete range $0 \leqq \alpha \leqq \infty$. In turn, the torsional rigidity for this configuration is independent of the tangential shear stiffness in this range and is given by (1.2).

We apply the stability results given in Theorems 7.1 and 7.2 to investigate the existence of isoperimetric inequalities for the extreme cases $\alpha = 0$ and $\alpha = \infty$. The first result is an isoperimetric inequality for the extreme case when there is a loss in the adhesion tangential to the fiber.

**Theorem 7.3** (Isoperimetric inequality for the case $\alpha = 0$). *When there is no adhesion tangential to the fibers, then of all fiber-reinforced shafts with given cross-sectional area $\pi R^2$ and given joint cross-sectional area of fibers $\pi a^2$, the shaft with circular cross section reinforced with a concentric circular fiber has the maximum torsional rigidity, i.e.,*

$$\mathrm{T}(A, \Omega, 0) \leqq \left(\tfrac{\pi}{2}\mathrm{G_m}(R^4 - a^4) + \tfrac{\pi}{2}\mathrm{G_f}a^4\right). \tag{7.17}$$

We start with a heuristic proof of this theorem based on the stability result given in Theorem 7.1. We observe that as $\alpha$ tends to zero, the parameter $\mathrm{R}_{\mathrm{cr}}^{-1}$ tends to zero. Thus the class of admissible fiber cross sections treated in the hypothesis of Theorem 1.6 increases to include all fiber cross sections in the $\alpha = 0$ limit. Applying Theorem 7.1 to inequality (1.12) gives (7.17) for all fiber cross sections. We now give a rigorous proof based on the formulation of the torsional rigidity as given by equation (7.8). Integration by parts in (7.8) and application of equations (7.5), (7.6), and (7.7) shows that

$$\mathrm{T}(A, \Omega, 0) = \left(2\sum_{i=1}^{N} c_i|\Sigma_i| + 2\int_{\Omega\setminus A} \Phi\, dx + 2\sum_{i=1}^{N}\int_{\Sigma_i} (\Phi - c_i)\, dx\right). \tag{7.18}$$

Here $2\sum_{i=1}^{N} c_i|\Sigma_i| + 2\int_{\Omega\setminus A} \Phi dx$ is the torsional rigidity of the multiply connected domain $\Omega \setminus A$ filled with material of shear stiffness $\mathrm{G_m}$, and $2\int_{\Sigma_i}(\Phi - c_i)dx$ is the torsional rigidity of the $i^{\text{th}}$ fiber made from material of shear stiffness $\mathrm{G_f}$. Application of POLYA'a [11] result to each fiber gives

$$2\sum_{i=1}^{N}\int_{\Sigma_i} (\Phi - c_i)dx \leqq \sum_{i=1}^{N} \tfrac{\pi}{2}\mathrm{G_f}a_i^4. \tag{7.19}$$

Here the area of each cross section $\Sigma_i$ is $\pi a_i^2$. Application of the of the POLYA & WEINSTEIN [12] result for the torsional rigidity for shafts with multiply connected cross section gives

$$2\sum_{i=1}^{N} c_i|\Sigma_i| + 2\int_{\Omega\setminus A} \Phi dx \leqq \tfrac{\pi}{2}\mathrm{G_m}(R^4 - a^4), \tag{7.20}$$

where $\pi R^2$ is the cross-sectional area of the shaft and $\pi a^2$ is the joint area of the fibers. Noting that $\pi a^2 = \pi \sum_{i=1}^{N} a_i^2$ we have $a^4 \geqq \sum_{i=1}^{N} a_i^4$ as before. Collecting our results we discover that

$$\mathrm{T}(A, \Omega, 0) \leqq \left(\tfrac{\pi}{2}\mathrm{G_m}(R^4 - a^4) + \tfrac{\pi}{2}\mathrm{G_f}a^4\right), \tag{7.21}$$

and Theorem 7.3 follows.

We show that the isoperimetric inequality given in Theorem 1.6 does not persist in the perfect bonding limit. To see this we consider the case of a shaft with circular cross section reinforced with a single fiber of circular cross section. From Theorem 1.1 we have that the concentric fiber and shaft configuration is strictly suboptimal when the fiber radius lies above $\mathrm{R_{cr}}$. Observing that $\mathrm{R_{cr}}$ tends to zero as $\alpha$ tends to infinity we invoke Theorem 7.2 and pass to the limit in inequality (5.1) of Theorem 5.1 to find that

$$\mathrm{T}(D_{\breve{x}}(a), D_0(R), \infty) > \left(\tfrac{\pi}{2}\mathrm{G_m}(R^4 - a^4) + \tfrac{\pi}{2}\mathrm{G_f}a^4\right) \tag{7.22}$$

for all fibers with centers $\breve{x} \neq 0$. Thus for $\alpha = \infty$ the concentric circular fiber and shaft configuration is suboptimal for any fiber radius $a$. This example shows that an isoperimetric inequality of the kind given in Theorem 1.6 does not hold in the limit of perfect bonding.

Similar remarks hold for Theorem 1.7 in the extreme cases $\alpha = 0$ and $\alpha = \infty$. Inequality (7.22) shows that there is no upper bound of the kind given by Theorem 1.7 in the $\alpha = \infty$ case. On the other hand when there is complete debonding in the directions tangential to the fiber-matrix interface we have

**Theorem 7.4** (Optimal design theorem for the $\alpha = 0$ case). *Consider any fiber-reinforced shaft with given cross-sectional area $\pi R^2$ and N fibers $\Sigma_1, \Sigma_2, \ldots, \Sigma_N$. If there is no adhesion tangential to the fiber-matrix interface, then the torsional rigidity is less than or equal to that of the shaft with circular cross section reinforced with a concentric circular fiber with radius $\breve{a}$ given by*

$$\pi\breve{a}^2 = \sqrt{|\Sigma_1|^2 + |\Sigma_2|^2 + \cdots |\Sigma_N|^2}, \tag{7.23}$$

*i.e.,*

$$\mathrm{T}(A, \Omega, 0) \leqq \left(\tfrac{\pi}{2}\mathrm{G_m}(R^4 - \breve{a}^4) + \tfrac{\pi}{2}\mathrm{G_f}\breve{a}^4\right). \tag{7.24}$$

To prove Thorem 7.4 we return to inequality (7.19) and choose $\breve{a}$ such that $\breve{a}^4 = \Sigma_{i=1}^{N} a_i^4$. It is evident that $\pi\breve{a}^2 \leqq \pi a^2$ where $\pi a^2$ is the joint fiber cross-sectional area, and the theorem follows immediately from (7.18), and the substitutions of $\breve{a}$ for $\Sigma_{i=1}^{N} a_i^4$ in (7.19) and $\breve{a}$ for $a$ in (7.20).

## 8. Maximum Torsional Rigidity in the Presence of Partial Adhesion Normal to the Fiber-Matrix Interface

In this Section we allow for partial adhesion in the direction normal to the fiber-matrix interface as well as in the tangential direction. We recall that $u_n^f$, $u_n^m$ denote the displacements normal to the fiber-matrix interface evaluated on the fiber and matrix sides of the interface respectively. We suppose that there is no interpenetration of material points on the interface, i.e., $u_n^{\mathrm{f}} - u_n^{\mathrm{m}} \leqq 0$. We replace the perfect adhesion condition normal to the fiber boundary given by (2.2) with the partial adhesion conditions:

$$\text{If } \; u_n^{\mathrm{f}} - u_n^{\mathrm{m}} = 0, \quad \text{then } \sigma_n \leqq 0; \tag{8.1}$$

$$\text{if } \; u_n^{\mathrm{f}} - u_n^{\mathrm{m}} < 0, \quad \text{then } \sigma_n = -\gamma(u_n^{\mathrm{f}} - u_n^{\mathrm{m}}). \tag{8.2}$$

Here $\gamma$ has units of shear stiffness per unit length. These conditions do not allow for interpenetration at the interface but do allow for material points on either side of the interface to separate, their relatave displacements being proportional to the normal traction. Condition (8.1) shows that the normal traction is directed into the fiber when points on either side of the interface are in contact. When material points are separated, (8.2) shows that the normal traction is directed away from the fiber. The interfacial conditions (8.1), (8.2) together with the conditions on the relative tangential displacement given by (2.4) are similar to those given by ACHENBACH & ZHU [2]. These interface conditions, together with the torsion boundary conditions and the equlibrium equation given in Section 2, deliver a boundary-value problem that uniquely determines the elastic displacement inside the shaft. This is proved in the second section of the Appendix. Inspection shows that the displacement given by (2.5) and (2.6) satisfies the interfacial transmission conditions (8.1) and (8.2). Indeed we have

$$u_n^{\mathrm{f}} - u_n^{\mathrm{m}} = 0, \quad \sigma_n = 0 \tag{8.3}$$

on every interface. The associated warping function is precisely the one obtained in Section 2. Thus, the displacement inside a shaft subject to torsion, does not depend on whether there is perfect or partial adhesion in the direction normal to the fiber-matrix interface, i.e, whether condition (2.2) or conditions (8.1), (8.2) hold. It is evident that the same holds true for the torsional rigidity, and we state

**Theorem 8.1.** *Theorems 1.1 through 1.7 hold true for the torsional rigidity, when in addition to partial adhesion in the directions tangential to the fiber-matrix interface, there is partial adhesion in the direction normal to the fiber-matrix interface as described by equations* (8.1) *and* (8.2).

## A. Appendix

We provide a short outline of the proof of existence of solution to the torsion boundary-value problem. The methods used are standard and can be found in the work of DUVAUT & LIONS [3]. Existence follows from the variational formulation of the problem given by

$$\min_{v \in V} \mathrm{F}(\boldsymbol{v}) \tag{A.1}$$

where

$$\mathrm{F}(\boldsymbol{v}) = \int_{B\setminus\Gamma} \mathscr{C}(x)e(\boldsymbol{v}) : e(\boldsymbol{v})\, dx + \alpha \int_{\Gamma} ([v_\tau])^2 ds, \tag{A.2}$$

$$V = \Big\{ \boldsymbol{v} \in H^1(B \setminus \Gamma)^3 |\ [\boldsymbol{v} \cdot \boldsymbol{n}]_\Gamma = 0, \quad v_1 = v_2 = 0 \text{ on } x_3 = 0, \\ v_1 = -\theta h x_2, \quad v_2 = \theta h x_1 \text{ on } x_3 = h, \int_B v_3\, dx = 0 \Big\}. \tag{A.3}$$

Here $\Gamma$ represents the fiber-matrix interface, $B$ is the shaft domain given by $B = \{(x_1, x_2) \in \Omega, 0 < x_3 < h\}$. Standard continuity properties of the trace operator show that $\mathrm{F}(\boldsymbol{v})$ is lower semicontinuous with respect to weak convergence in $H^1(B \setminus \Gamma)^3$. Thus we need only establish the coercivity of $\mathrm{F}(\boldsymbol{v})$ to show existence of a minimizer.

We establish the necessary coercivity property given by:

**Theorem A.1.** *There exist positive constants a and b such that*

$$a\|\boldsymbol{v}\| - b \leqq \mathrm{F}(\boldsymbol{v}) \quad \textit{for all elements } \boldsymbol{v} \textit{ in } V. \tag{A.4}$$

Here $\|\cdot\|$ is the $H^1$ norm on the domain $B \setminus \Gamma$.

We let $\Phi$ be any function in $H^1(B \setminus \Gamma)^3$ for which $[\Phi \cdot \boldsymbol{n}] = 0$, $\Phi_1 = \Phi_2 = 0$ on $x_3 = 0$ and $\Phi_1 = -\theta h x_2$, $\Phi_2 = \theta h x_1$ on $x_3 = h$, and $\int_B \Phi_3 dx = 0$. Then any element $\boldsymbol{v}$ in the space $V$ can be written as $\boldsymbol{v} = \boldsymbol{w} + \Phi$ where $\boldsymbol{w}$ lies in the closed subspace $V_0$ of $H^1(B \setminus \Gamma)^3$ given by

$$V_0 = \Big\{ \boldsymbol{w} \in H^1(B/\Gamma)^3 \mid [\boldsymbol{w} \cdot \boldsymbol{n}]_\Gamma = 0, \quad w_1 = w_2 = 0 \text{ on } x_3 = 0, \\ w_1 = w_2 = 0, \text{ on } x_3 = h, \int_B w_3 dx = 0 \Big\}. \tag{A.5}$$

We suppose that there exists a constant $K > 0$, such that

$$\|\boldsymbol{w}\| \leqq K F(\boldsymbol{w}) \quad \text{for all } \boldsymbol{w} \text{ in } V_0. \tag{A.6}$$

This inequality is established in the sequel. Writing any element $\boldsymbol{v}$ of $V$ as $\boldsymbol{v} = \boldsymbol{w} + \Phi$ and estimating gives

$$\|\boldsymbol{v}\| \leqq \|\boldsymbol{w}\| + \|\Phi\| \leqq K F(\boldsymbol{w}) + \|\Phi\|. \tag{A.7}$$

Noting that

$$F(\boldsymbol{w}) = F(\boldsymbol{v} - \Phi) \leqq 2(F(\boldsymbol{v}) + F(\Phi)) \tag{A.8}$$

we apply (A.7) to find that

$$\frac{1}{2K}\|v\| - \frac{1}{2K}\|\Phi\| - F(\Phi) \leqq F(v), \tag{A.9}$$

and the coercivity follows. It remains to show (A.6). The proof of this inequality follows arguments identical to those given in the work of LENE & LEGUILLION [4]. We introduce the form $|||v||| = \int_B |e(v)|^2 dx + \int_{B_c} |v|^2 dx + \int_\Gamma ([v_\tau])^2 ds$, where $B_c$ is the domain exterior to the fibers. LENE & LEGUILLION [4] show that $|||\cdot|||$ is equivalent to the standard $H^1$ norm. To complete the proof we show that there exists a constant $C > 0$

$$|||w||| < CF(w) \quad \text{such that for any } w \text{ in } V_0. \tag{A.10}$$

This is equivalent to showing the Poincaré-like inequality $\int_{B_c} |w|^2 dx < CF(w)$. To prove this we argue by contradiction. Suppose for each positive integer $N$ there exists an element $w_N$ in $V_0$ such that

$$\int_{B_c} |w_N|^2 dx \geqq NF(w_N). \tag{A.11}$$

Set $u_N = w_N \big(\int_{B_c} |w_N|^2 dx\big)^{-1/2}$. Then $\int_{B_c} |u_N|^2 dx = 1$ and

$$N^{-1} \geqq \int_B \mathscr{C} e(u_N) : e(u_N)\, dx + \alpha \int_\Gamma ([u_{N_\tau}])^2 ds. \tag{A.12}$$

Since $\mathscr{C} > 0$, there exists a subsequence also denoted by $u_N$ converging weakly in $H^1$ to an element $u$ in $V_0$. By weak lower semicontinuity we have $[u \cdot n]_\Gamma = 0$, $[u \cdot \tau]_\Gamma = 0$, and $e(u) = 0$ in $B \backslash \Gamma$. Thus we conclude that $u = 0$ almost everywhere in $B$. However this contradicts the fact that $\int_{B_c} |u|^2 dx = 1$ and the Poincaré-like inequality is proved.

## B. Uniqueness Theorem for the Torsion Boundary-Value Problem

We establish uniqueness of solution when there is partial adhesion in both normal and tangential directions to the fiber-matrix interface. Uniqueness for the case of partial adhesion in the tangential direction and perfect bonding in the normal direction follows from the estimate (A.6) and arguments identical to the ones given below. We introduce the bilinear form defined on $H^1(B/\Gamma)^3$ given by

$$a(v, u) = \int_{B/\Gamma} \mathscr{C}(x) e(v) : e(u) dx + \alpha \int_\Gamma ([v_\tau] \cdot [u_\tau] ds + \gamma \int_\Gamma [v_n][u_n] ds, \tag{B.1}$$

and the convex set $K$ given by

$$K = \Big\{ v \in H^1(B/\Gamma)^3 |\, [v_n] \leqq 0 \text{ on } \Gamma,\ \ v_1 = v_2 = 0 \text{ on } x_3 = 0, \\ v_1 = -\theta h x_2,\ \ v_2 = \theta h x_1 \ \text{ on } x_3 = h, \text{ and } \int_B v_3 dx = 0 \Big\}. \tag{B.2}$$

The solution $\tilde{u}$ of the torsion boundary-value problem with partial interfacial adhesion in both tangential and normal directions satisfies the equivalent variational inequality

$$a(\tilde{u}, v - \tilde{u}) \geqq 0 \quad \text{for all } v \text{ in } K. \tag{B.3}$$

This follows easily from standard considerations; see, DUVAUT & LIONS [3]. Arguments identical to those of the last section show that there exsists a positive constant $C$ for which

$$a(w, w) \geqq C\|w\| \tag{B.4}$$

for all functions $w$ in the space $\mathscr{H}_0$ given by

$$\mathscr{H}_0 = \left\{ w \in H^1(B/\Gamma)^3 \mid \; w_1 = w_2 = 0 \text{ on } x_3 = 0, \right. \\ \left. w_1 = 0, w_2 = 0 \text{ on } x_3 = h, \int_B w_3 dx = 0 \right\}. \tag{B.5}$$

We suppose the existence of a second solution $u$ and set $v = u$ in (B.3) to obtain

$$a(\tilde{u}, u - \tilde{u}) \geqq 0. \tag{B.6}$$

Since $u$ is a solution, we choose $v = \tilde{u}$ to find

$$a(u, \tilde{u} - u) \geqq 0. \tag{B.7}$$

Adding inequalities (B.6) and (B.7) gives

$$-a(u - \tilde{u}, u - \tilde{u}) \geqq 0. \tag{B.8}$$

Since $u - \tilde{u}$ lies in $\mathscr{H}_0$, it is evident from (B.4) that $u - \tilde{u} = 0$ and uniqueness is established.

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