PROCEEDINGS
OF
THE ROYAL
SOCIETY A

Proc. R. Soc. A (2007) **463**, 3291–3309
doi:10.1098/rspa.2007.0070
Published online 2 October 2007

Bounds for the torsional rigidity of shafts with arbitrary cross-sections containing cylindrically orthotropic fibres or coated fibres

By Tungyang Chen[1,*] and Robert Lipton[2]

[1]*Department of Civil Engineering, National Cheng Kung University, Tainan 70101, Taiwan*
[2]*Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA*

In this paper we derive bounds for the torsional rigidity of a cylindrical shaft with arbitrary transverse cross-section containing a number of cylindrically orthotropic fibres or coated fibres. The exact upper and lower bounds depend on the constituent shear rigidities, the area fractions, the locations of the reinforcements as well as the geometric shape of the cross-sections. Specific bounds are derived for circular shafts, elliptical shafts and cross-sections of equilateral triangle. Simplified expressions are also deduced for reinforcements with isotropic constituents. We verify that when additional constraints between the constituent properties of the phases are fulfilled, the upper and lower bounds will coincide. In the latter case, the fibres or coated fibres become neutral under torsion and the bounds recover the previously known exact torsional rigidity.

Keywords: torsional rigidity; bounds; cylindrically orthotropic fibres

1. Introduction

Saint-Venant's torsion of cylindrical shafts has long been a subject of classical mechanics. The torsional rigidity depends in a complicated way on the geometric shape of the cross-section, constituent shear rigidities as well as on its microstructure. An exact characterization of the composite shafts often poses some mathematical difficulties. Recently, substantial advances have been made for torsion of composite shafts, made from two or more different materials. For example, Chen *et al.* (2002) found that a circular shaft filled with an assemblage of composite cylinders permits an exact determination of the torsional rigidity. Chen (2004) showed that, depending on the cross-sectional shape of the host shaft, the torsional rigidity of the composite shaft could be exactly determined if the composite cylinders are suitably multicoated. Ting *et al.* (2004) extended the concept and showed how to design a multicoated cylinder with cylindrically orthotropic constituents so that the composite shaft filled with an assemblage of multicoated cylinders can be exactly analysed. All these developments were indirect, based on a construction of neutral inclusions under torsion. A neutral

* Author for correspondence (tchen@mail.ncku.edu.tw).

Received 11 June 2007
Accepted 7 September 2007

inclusion in torsion was defined as a cylindrical inclusion which can be introduced into a homogeneous host shaft subjected to torsion, without disturbing the warping field in the host shaft. Neutral inclusions for Saint-Venant's torsion problem were constructed by Lipton (1998) for the case of imperfectly bonded inclusions placed inside a more compliant matrix. For this case, the neutral inclusions have a radius determined by the elastic moduli of the constituents and the stiffness of the imperfect interface. Other neutral inclusions in the presence of imperfect interface for three-dimensional elastic composites are constructed in Lipton & Vernescu (1995). For coated neutral inclusions, see the book of Milton (2002, ch. 7) for a detailed exposition of related references. However, exact formulae for the torsional rigidity are not usually available for more general fibre geometries and material properties. Therefore, it is also useful to have explicit but sharp estimates on the torsional rigidity for more general situations. In this paper we pursue this line of investigation with the goal of providing explicit and sharp bounds for the torsional rigidity that hold for a large class of fibre configurations.

The problem of bounding the torsional rigidity for a cylindrical shaft has long been a subject of fundamental interest in solid mechanics. For homogeneous shafts made of elastically isotropic material, de Saint-Venant (1856) proposed that among all cylindrical shafts with given cross-sectional area, circular shafts will give the greatest torsional rigidity. A rigorous proof of this proposition was made nearly one century after by Diaz & Weinstein (1948) and Polya (1948). Polya & Weinstein (1950) showed that among all multiply connected sections with given cross-sectional area and with given joint area of holes, the ring bounded by two concentric circles has the maximum torsional rigidity. Upper and lower bounds on torsional rigidity were also investigated by Payne & Weinberger (1961), Payne (1962) and some others. All these above results are mainly concerned with homogeneous cross-sections. For composite shafts, Alvino & Trombetti (1985) showed that circular cross-sections with a radially non-increasing arrangement of compliance will give the maximum torsional rigidity among all cross-sections with given cross-sectional area and fixed area fraction of the constituent phases. Lipton (1998, 1999) presented a variational approach to examine composite shafts with imperfect interfaces between fibre and matrix phases. For this case, the condition of the continuity of the warping displacement is relaxed and replaced with a 'spring-type' interface prevailing along the host shaft and the fibres. Optimal bounds and isoperimetric inequalities were introduced illustrating the dependence of the torsional rigidity on the degree of imperfect bonding together with shape and position of the fibres within the shaft.

More recently, Lipton & Chen (2004) developed a variational approach for bounding the torsional rigidity of shafts containing coated fibre reinforcements. The first set of results are reinforcement inequalities that provide explicit criteria that determine when the torsional rigidity of a particular coated fibre configuration lies above or below the torsional rigidity of the coated cylinder assemblage found by Chen *et al.* (2002). The torsional rigidity for this assemblage is identical to the torsional rigidity of a single-coated fibre of circular cross-section centred inside a shaft of circular cross-section. When additional (neutrality) conditions are fulfilled by the warping function inside the shaft then the upper and lower inequalities coincide and agree with the exact torsional rigidity found by Chen *et al.* (2002). The second set of results are geometric inequalities that show how the effective anti-plane shear rigidity and torsional rigidity of each coated fibre can be used to determine whether a particular configuration provides reinforcement above or below that of a homogeneous shaft containing no coated fibres. These are obtained using techniques based upon the solution of eigenvalue problems associated with each coated fibre. Amazingly, the lowest eigenvalue for these problems is given in terms of the effective anti-plane shear modulus of the coated fibre.

The present study focuses on the former set of problems addressed by Lipton & Chen (2004) but with emphasis on shafts with arbitrary cross-sections. The shaft contains a number of fibres or coated fibres, which can be arbitrarily positioned inside the shaft. The constitutive law for the reinforcements is allowed to be cylindrically orthotropic and the host medium, referred to as the matrix, is elastically isotropic. The elastic symmetry of cylindrical orthotropy is characterized by three different shear rigidities in the radial, circumferential and axial directions, respectively. Here under Saint-Venant's torsion, only the shear rigidities in the radial and circumferential directions will take effect. In this work, the choice of trial fields used in the variational principles is motivated by the fields inside the exactly solvable microgeometries for shafts with arbitrary cross-section introduced and discussed by Chen (2004). The exact upper and lower bounds depend on the constituent shear rigidities, the area fractions, the locations of the reinforcements as well as the geometric shape of the cross-section. The shape factor of the cross-section will explicitly enter into the expressions of the bounds. As demonstrated by Ting *et al.* (2004) and Chen & Wei (2005), these types of anisotropic materials provide more degrees of freedom allowing more opportunities for finding explicit formulae for the torsional rigidities. It is found that the expressions for the explicit bounds on the torsional rigidity for cross-sections containing non-coated fibres are distinct from that of cross-sections containing coated fibres. The former involves a single summation contributing from each non-coated fibres, while the latter involves a double summation, which in addition to the contribution of each coated fibre, also involves a summation relevant to the shape effect of the host shaft.

The paper is organized as follows. We first provide a brief outline of the bounds derived in §2. In §3 we derive the upper bounds for the torsional rigidity. The solutions are formulated in terms of virtual warping displacement. Derivations for the lower bounds given in §4 are based on constructing virtual stress potential. In §5, specific bounds are derived for circular shafts, elliptical shafts and cross-sections of equilateral triangle. Simplified expressions are also provided for phases with isotropic constituents. Lastly, a generalization to shafts containing many multicoated fibres is envisaged.

2. Inequalities on the torsional rigidity

We consider the Saint-Venant torsion of cylindrical shafts. The shafts are made of a homogeneous isotropic matrix containing N cylindrically orthotropic fibres. The fibres have circular cross-section and the radius of the ith fibre, $i=1, \ldots, N$ is denoted by a_i. The fibres may also be coated by a shell of cylindrically orthotropic material of uniform thickness, then, in this situation, the outer radius of the ith coating will be denoted by b_i. Each coated fibre may have different constituent materials with different area fractions. The area fraction of the fibre for the ith coated fibre is denoted by ν_i and $\nu_i = a_i^2/b_i^2$. The matrix has the isotropic shear modulus denoted by μ_{m}, while the fibres and the coatings are cylindrically orthotropic.

In order to describe the cylindrical orthotropy, let us now consider a cylindrical inclusion, which could be a coated or a non-coated fibre. Suppose that the centre of the inclusion is located at a certain point O_i inside the cross-section Ω. The position O_i will be designated as $\hat{\boldsymbol{x}}_i = (\hat{x}_i, \hat{y}_i)$ relative to the origin of $\boldsymbol{x}$. Consider a second coordinate system obtained from the first coordinate by a translation without a change in orientation. We denote this new coordinate system by (X, Y, z), in which $x = X + \hat{x}_i$ and $y = Y + \hat{y}_i$. We introduce the polar coordinates (r, θ, z) centred at O_i with $X = r\cos\theta$ and $Y = r\sin\theta$. The unit vectors in the radial and tangential directions are denoted by $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$, respectively. The rigidity tensor inside cylindrically orthotropic fibre and coating is of the form

$$\boldsymbol{\mu} = \mu_r(X, Y)\boldsymbol{e}_r\boldsymbol{e}_r + \mu_\theta(X, Y)\boldsymbol{e}_\theta\boldsymbol{e}_\theta. \tag{2.1}$$

For (X, Y) in the ith fibre $\mu_r(X, Y) = \mu_r^{i(\mathrm{f})}$, $\mu_\theta(X, Y) = \mu_\theta^{i(\mathrm{f})}$ and for (X, Y) in the corresponding coating $\mu_r(X, Y) = \mu_r^{i(\mathrm{c})}$ and $\mu_\theta(X, Y) = \mu_\theta^{i(\mathrm{c})}$. The shaft cross-section, denoted by Ω is simply connected and its outer boundary is denoted by $\partial\Omega$. The domain occupied by the ith fibre and its coating is denoted by $\Sigma_i = \{X^2 + Y^2 \le b_i^2\}$. The union of all the fibres is denoted by Σ_{f} and that of the coatings is by Σ_{c}. The remaining part of the cross-section containing matrix material is denoted by Ω_{m} and thus $\Omega = \Omega_{\mathrm{m}} \bigcup \Sigma_{\mathrm{f}} \bigcup \Sigma_{\mathrm{c}}$. At the interfaces between any two adjacent phases, we assume that they are perfectly bonded.

Instead of directly solving for the warping and associated stress fields inside the composite shaft, we introduce variational principles and derive sharp upper and lower bounds for the torsional rigidity of the composite shaft. The bounds given here depend on the elastic properties of the constituents, the cross-sectional shape of Ω, the area fractions ν_i as well as the position of the reinforcements $\hat{\boldsymbol{x}}_i$. We describe the variational principles used to derive the bounds for the torsional rigidity. The torsional rigidity for a shaft with cross-section Ω containing N fibres or coated fibres is denoted by $\mathcal{T}_N(\Omega)$. The first variational principle is given in terms of virtual warping functions $\tilde{w}$ that are square integrable with square integrable gradients. It is given by

$$\mathcal{T}_N(\Omega) = \min_{\tilde{w}} \left\{ \int_\Omega \boldsymbol{\mu}(\boldsymbol{x})(\nabla\tilde{w} + \boldsymbol{x}^\perp)\cdot(\nabla\tilde{w} + \boldsymbol{x}^\perp)\,\mathrm{d}\boldsymbol{x} \right\}. \tag{2.2}$$

Here $\boldsymbol{x}^\perp$ is the anti-clockwise rotation of $\boldsymbol{x}$ through $\pi/2$ radians. Note that the piecewise constant shear matrix is $\mu_{\mathrm{m}}\boldsymbol{I}$ in the matrix and is (2.1) for the coated fibres. This variational principle is minimized by the actual warping function inside the shaft.

The second variational principle is given in terms of virtual stress potentials $\tilde{\varphi}$ that vanish on the boundary of the shaft cross-section $\partial\Omega$ and are square integrable and have square integrable gradients. The second variational principle for the torsional rigidity is given by

$$\mathcal{T}_N(\Omega) = \max_{\tilde{\varphi}} \left\{ 4\int_\Omega \tilde{\varphi}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \left(\frac{\boldsymbol{\mu}(\boldsymbol{x})}{\det \boldsymbol{\mu}(\boldsymbol{x})} \nabla\tilde{\varphi} \right)\cdot\nabla\tilde{\varphi}\,\mathrm{d}\boldsymbol{x} \right\}. \tag{2.3}$$

This variational principle is maximized by the actual stress potential inside the shaft.

We start by displaying bounds on the torsional rigidity for reinforcements that are non-coated cylindrically orthotropic fibres. For a system of N cylindrically orthotropic fibres located at the points $\hat{\boldsymbol{x}}_i$, $i=1,\ldots,N$, we construct upper and lower bounds for the torsional rigidity. The main result is stated in proposition 2.1.

Proposition 2.1. *For a shaft with arbitrary cross-section Ω containing N cylindrically orthotropic fibres with radii a_i, we find that the torsional rigidity of the reinforced shaft $\mathcal{T}_N(\Omega)$ is bounded between the quantities*

$$\mathcal{A}^N + \pi \sum_{i=1}^{N} \Xi^i \frac{\mu_{\mathrm{m}}}{\mu_{\mathrm{G}}^i}\left(\mu_{\mathrm{G}}^i - \mu_{\mathrm{m}}\right) \le \mathcal{T}_N(\Omega) \le \mathcal{A}^N + \pi \sum_{i=1}^{N} \Xi^i \left(\mu_{\mathrm{G}}^i - \mu_{\mathrm{m}}\right), \tag{2.4}$$

where $\mathcal{A}^N$ and Ξ^i are defined by

$$\mathcal{A}^N = \mathcal{T}^{\mathrm{m}}(\Omega) + \frac{\pi}{2}\sum_{i=1}^{N}\left(\mu_\theta^{i(\mathrm{f})} - \mu_{\mathrm{m}}\right) a_i^4 \quad \textit{and} \tag{2.5}$$

$$\Xi^i = \sum_{n=1}^{S} n\left(\left(\alpha_n^i\right)^2 + \left(\beta_n^i\right)^2\right) a_i^{2n}. \tag{2.6}$$

$\mathcal{T}^{\mathrm{m}}(\Omega)$ is the torsional rigidity of the homogeneous shaft with cross-section Ω and μ_{G}^i is the geometric mean of the shear rigidity defined by

$$\mu_{\mathrm{G}}^i = \sqrt{\mu_r^{i(\mathrm{f})}\mu_\theta^{i(\mathrm{f})}}. \tag{2.7}$$

The upper and lower bounds agree when $\mu_{\mathrm{m}} = \mu_{\mathrm{G}}^i$, $i=1,\ldots,N$.

Note that the position of the fibres, $\hat{\boldsymbol{x}}_i$, $i=1,\ldots,N$ and the cross-sectional shape of Ω are expressed through the coefficients S, α_n^i and β_n^i. The parameter S could be a finite or an infinite value depending on the cross-sectional shape of the shaft. For instance, for a circular shaft we have $S=1$. Detailed expressions for the coefficients S, α_n^i and β_n^i for a few simple cross-sectional shapes were given by Chen (2004) and are recorded in §3 for completeness.

Next we suppose that the reinforcements are coated fibres. The variational bounds on the torsional rigidity are given in proposition 2.2.

Proposition 2.2. *For a shaft with arbitrary cross-section Ω containing N cylindrically orthotropic coated fibres, the torsional rigidity of the reinforced shaft $\mathcal{T}_N(\Omega)$ is bounded between the quantities*

$$\begin{aligned}\mathcal{B}^N + \pi \sum_{i=1}^{N}\sum_{n=1}^{S} \Upsilon^i(n) \frac{\mu_{\mathrm{m}}}{\mu_{\mathrm{CCA}}^i(n)}\left(\mu_{\mathrm{CCA}}^i(n) - \mu_{\mathrm{m}}\right) \le \mathcal{T}_N(\Omega) \le \mathcal{B}^N \\ + \pi \sum_{i=1}^{N}\sum_{n=1}^{S} \Upsilon^i(n)\left(\mu_{\mathrm{CCA}}^i(n) - \mu_{\mathrm{m}}\right),\end{aligned} \tag{2.8}$$

where $\mathcal{B}^N$ and Ξ^i are given by

$$\mathcal{B}^N = \mathcal{T}^{\mathrm{m}}(\Omega) + \frac{\pi}{2}\sum_{i=1}^{N}\left(\mu_\theta^{i(\mathrm{c})}\left(b_i^4 - a_i^4\right) + \mu_\theta^{i(\mathrm{f})} a_i^4 - \mu_{\mathrm{m}} b_i^4\right), \tag{2.9}$$

$$\Upsilon^i(n) = n\left(\left(\alpha_n^i\right)^2 + \left(\beta_n^i\right)^2\right) b_i^{2n}, \tag{2.10}$$

and $\mu^i_{\rm CCA}(n)$ *is the shear rigidity defined by*

$$\mu^i_{\rm CCA}(n) = \mu_{\rm G}^{i(\rm c)} \frac{\left[(g^i+1)+\nu_i^{n\lambda_{\rm c}}(g^i-1)\right]}{\left[(g^i+1)-\nu_i^{n\lambda_{\rm c}}(g^i-1)\right]}, \tag{2.11}$$

with

$$\lambda_{\rm c} = \sqrt{\mu_\theta^{(\rm c)}/\mu_r^{(\rm c)}} \quad \textit{and} \quad g^i = \frac{\mu_{\rm G}^{i(\rm f)}}{\mu_{\rm G}^{i(\rm c)}} = \frac{\sqrt{\mu_r^{i(\rm f)}\mu_\theta^{i(\rm f)}}}{\sqrt{\mu_r^{i(\rm c)}\mu_\theta^{i(\rm c)}}}. \tag{2.12}$$

We mention that the value $\mu^i_{\rm CCA}(n)$, which depends on n with $n=1,\dots,S$, is the effective anti-plane shear rigidity of an assemblage of coated cylinders, with cylindrically orthotropic constituents, under displacement boundary conditions of order n. The expression $\mu^i_{\rm CCA}$ differs intrinsically with that of non-coated fibres in which $\mu^i_{\rm G}$ does not vary with n. It is seen that the upper and lower bounds will coincide when $\mu_{\rm m}=\mu^i_{\rm CCA}(n)$, for all $n=1,\dots,S$ and $i=1,\dots,N$.

3. Upper bounds on the torsional rigidity for shafts of arbitrary cross-section reinforced with coated or uncoated fibres

In this section we develop trial warping functions for cylindrical shafts of arbitrary cross-section Ω containing N circular cylindrically orthotropic fibres or coated fibres. The trial fields will be substituted into the variational principle (2.2) to deliver the upper bound given in propositions 2.1 and 2.2.

First, consider a homogeneous shaft filled with pure matrix material. For a fixed Cartesian coordinate system centred at O, the associated warping displacement, denoted by $w_{\rm m}(x, y)$, is harmonic inside Ω and on $\partial\Omega$

$$\frac{\partial w_{\rm m}}{\partial n} = -\boldsymbol{x}^\perp\cdot\boldsymbol{n}\Big|_{\partial\Omega}. \tag{3.1}$$

The trial warping function $\tilde{w}$ is taken to be equal to $w_{\rm m}$ outside the inclusions. The inclusions can be either coated or uncoated fibres. Inside the coated or uncoated fibre centred at $\hat{\boldsymbol{x}}_i$, the trial field $\tilde{w}$ is the solution of the boundary-value problem given by $\tilde{w}=w_{\rm m}$ on the inclusion matrix boundary, and inside the coating and fibre $\tilde{w}$ satisfies

$$\nabla\cdot(\boldsymbol{\mu}(\boldsymbol{x})\nabla\tilde{w}) = 0, \tag{3.2}$$

where $\boldsymbol{\mu}(\boldsymbol{x})$ is given by (2.1). The transmission conditions across the coating–fibre interface J are

$$[\tilde{w}]\Big|_J = 0, \qquad \left[\boldsymbol{\mu}(\boldsymbol{x})(\nabla\tilde{w}+\boldsymbol{x}^\perp)\cdot\boldsymbol{e}_r\right]\Big|_J = 0, \tag{3.3}$$

where $[\cdot]|_J$ denotes the jump of a quantity across J. We note further that if the trial satisfies the extra traction continuity condition at the interface between Σ_i and the matrix given by

$$\boldsymbol{\mu}(\boldsymbol{x})(\nabla\tilde{w}+\boldsymbol{x}^\perp)\cdot\boldsymbol{e}_r = \mu_{\rm m}(\nabla\tilde{w}+\boldsymbol{x}^\perp)\cdot\boldsymbol{e}_r, \tag{3.4}$$

where the l.h.s. is evaluated on the interface just inside Σ_i and the r.h.s. is evaluated on the matrix side of the interface, then the trial is the actual warping function inside the composite shaft.

Substitution of this trial field into the variational principle gives the upper bound

$$U_N = \int_{\Omega_\mathrm{m}} \mu_\mathrm{m}(\nabla\tilde{w} + \boldsymbol{x}^\perp)\cdot(\nabla\tilde{w} + \boldsymbol{x}^\perp)\mathrm{d}\boldsymbol{x} + \sum_{i=1}^{n}\int_{\Sigma_i} \boldsymbol{\mu}(\boldsymbol{x})(\nabla\tilde{w} + \boldsymbol{x}^\perp)\cdot(\nabla\tilde{w} + \boldsymbol{x}^\perp)\mathrm{d}\boldsymbol{x}. \tag{3.5}$$

Adding and subtracting the torsional rigidity of the cross-section filled with pure matrix material $\mathcal{T}^\mathrm{m}(\Omega)$ gives

$$\begin{aligned} U_N = \mathcal{T}^\mathrm{m}(\Omega) &+ \sum_{i=1}^{N}\int_{\Sigma_i} \boldsymbol{\mu}(\boldsymbol{x})(\nabla\tilde{w} + \boldsymbol{x}^\perp)\cdot(\nabla\tilde{w} + \boldsymbol{x}^\perp)\mathrm{d}\boldsymbol{x} \\ &- \sum_{i=1}^{N}\int_{\Sigma_i} \mu_\mathrm{m}(\nabla w_\mathrm{m} + \boldsymbol{x}^\perp)\cdot(\nabla w_\mathrm{m} + \boldsymbol{x}^\perp)\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{3.6}$$

The expressions for the upper bounds (2.4) and (2.8) on the torsional rigidity are obtained by deriving explicit expressions for the quantities

$$\begin{aligned} H_i = &\int_{\Sigma_i} \boldsymbol{\mu}(\boldsymbol{x})(\nabla\tilde{w} + \boldsymbol{x}^\perp)\cdot(\nabla\tilde{w} + \boldsymbol{x}^\perp)\mathrm{d}\boldsymbol{x} \\ &- \int_{\Sigma_i} \mu_\mathrm{m}(\nabla w_\mathrm{m} + \boldsymbol{x}^\perp)\cdot(\nabla w_\mathrm{m} + \boldsymbol{x}^\perp)\mathrm{d}\boldsymbol{x}, \quad i = 1, \ldots, N. \end{aligned} \tag{3.7}$$

To calculate these quantities associated with Σ_i for coated or non-coated fibres, it is convenient to translate the coordinates to the location of the centre of the ith fibre O_i. The specific transformation of the warping function under this change of coordinates is pointed out in Chen *et al.* (2002). In particular, the warping displacement for a cross-section filled with pure matrix material in the new coordinates $w_\mathrm{m}|_{O_i}$ is related to the warping function in the old coordinates by $w_\mathrm{m}|_{O_i} = w_\mathrm{m}(X+\hat{x}_i, Y+\hat{y}_i) - \hat{y}_i X + \hat{x}_i Y$. Since the warping field is harmonic inside Ω, the function $w_\mathrm{m}|_{O_i}$ can be expanded about the point O_i in the form of trigonometric series with a sufficient number of terms S (Chen 2004),

$$w_\mathrm{m}\Big|_{O_i} = \sum_{n=0}^{S} r^n(\alpha_n \cos n\theta + \beta_n \sin n\theta), \tag{3.8}$$

where the coefficients α_n and β_n are constants. For example, for a circular Ω, the non-zero coefficients of α_n and β_n are

$$\alpha_1 = -\hat{y}_i \quad \text{and} \quad \beta_1 = \hat{x}_i. \tag{3.9}$$

This implies that $S=1$. For an elliptical Ω, the non-vanishing coefficients of α_n and β_n are given by

$$\alpha_1 = -(\kappa+1)\hat{y}_i, \qquad \beta_1 = -(\kappa-1)\hat{x}_i \quad \text{and} \quad \beta_2 = -\kappa/2, \tag{3.10}$$

where $\kappa = (A^2 - B^2)/(A^2 + B^2)$, and A and B are the semi-axes of the ellipse. In this case, $S=2$. Also, if Ω is an equilateral triangle with side length l, then

$$\alpha_1 = 2\sqrt{3}\hat{x}_i\hat{y}_i/l - \hat{y}_i, \qquad \alpha_2 = \sqrt{3}\hat{y}_i/l, \qquad \beta_1 = \sqrt{3}(\hat{x}_i^2 - \hat{y}_i^2)/l + \hat{x}_i,$$
$$\beta_2 = \sqrt{3}\hat{x}_i/l, \qquad \beta_3 = 1/\sqrt{3}l \tag{3.11}$$

and $S=3$. When the cross-section Ω becomes irregular, e.g. a rectangle, the number of S will become larger or even unbounded.

Next we consider the trial warping displacement inside Σ_i associated with the local coordinates centred at O_i. The trial warping function inside Σ_i for the local coordinate system is denoted by $\tilde{w}_i$ and is related to the trial warping function inside Σ_i in the old coordinates by $\tilde{w}_i = \tilde{w}(X + \hat{x}_i, Y + \hat{y}_i) - \hat{y}_i X + \hat{x}_i Y$. We let $\tilde{w}_i^{\mathrm{f}}$ denote the trial inside the fibre and $\tilde{w}_i^{\mathrm{c}}$ the trial inside the coating. In the local polar coordinate system, $\tilde{w}_i$ is the solution of

$$\frac{\partial^2 \tilde{w}_i^k}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{w}_i^k}{\partial r} + \frac{\lambda_k^2}{r^2}\frac{\partial^2 \tilde{w}_i^k}{\partial \theta^2} = 0, \quad \text{with} \quad \lambda_k = \sqrt{\mu_\theta^{(k)}/\mu_r^{(k)}}, \tag{3.12}$$

for $k=\mathrm{f}$ and c. At the interface J between the fibre and the coating, we require that

$$[\tilde{w}_i]\Big|_J = 0 \quad \text{and} \quad \left[\mu_r \frac{\partial \tilde{w}_i}{\partial r}\right]\Big|_J = 0 \tag{3.13}$$

and at the interface between Σ_i and the matrix we have

$$\tilde{w}_i = w_{\mathrm{m}}|_{O_i}. \tag{3.14}$$

The quantities H_i defined by (3.7) can now be expressed in terms of the local coordinate representation of the trial warping functions and are given by

$$H_i = \int_{\Sigma_i} \left\{ \mu_r(r\cos\theta, r\sin\theta)\left(\frac{\partial \tilde{w}_i}{\partial r}\right)^2 + \mu_\theta(r\cos\theta, r\sin\theta)\left(\frac{1}{r}\frac{\partial \tilde{w}_i}{\partial \theta} + r\right)^2 \right\} r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$- \int_{\Sigma_i} \mu_{\mathrm{m}} \left(\left(\frac{\partial w_{\mathrm{m}}|_{O_i}}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial w_{\mathrm{m}}|_{O_i}}{\partial \theta} + r\right)^2 \right) r\,\mathrm{d}r\,\mathrm{d}\theta, \quad i = 1, \ldots, N. \tag{3.15}$$

In the following subsections we evaluate H_i for both non-coated and coated fibres.

(*a*) *Reinforcements are cylindrically orthotropic fibres*

We first treat the case when the reinforcements are cylindrically orthotropic non-coated fibres. Here there is no coating phase and Σ_i consists of a cylindrically orthotropic fibre. The trial function $\tilde{w}_i^{\mathrm{f}}$ inside the fibre is required to satisfy (3.12) and the boundary condition (3.14) is

$$\tilde{w}_i^{\mathrm{f}} = w_{\mathrm{m}}|_{O_i}, \tag{3.16}$$

on the boundary of Σ_i, i.e. $r = a_i$. We solve the boundary-value problem within each fibre to obtain the explicit formula for $\tilde{w}_i^{\mathrm{f}}$. Since $\tilde{w}_i^{\mathrm{f}}$ satisfies (3.12), it can be represented as

$$\tilde{w}_i^{\mathrm{f}} = \sum_{n=0}^{S} r^{n\lambda}(A_n \cos n\theta + B_n \sin n\theta), \quad \text{for } r \le a_i. \tag{3.17}$$

The coefficients A_n and B_n are determined by (3.16) and are given by

$$A_n = \alpha_n a_i^{n(1-\lambda)} \quad \text{and} \quad B_n = \beta_n a_i^{n(1-\lambda)}. \tag{3.18}$$

Substitution of $w_{\mathrm{m}}|_{O_i}$ (see (3.8)) and $\tilde{w}_i = \tilde{w}_i^{\mathrm{f}}$ into (3.15) and (3.6) delivers the upper bound given in (2.4).

We point out that if the traction is continuous across the fibre–matrix interface (see (3.4)), then we have the additional condition for the warping functions given by

$$\mu_r^i \frac{\partial \tilde{w}_i^{\mathrm{f}}}{\partial r}\bigg|_{r=a_i} = \mu_{\mathrm{m}} \frac{\partial w_{\mathrm{m}}|_{O_i}}{\partial r}\bigg|_{r=a_i}, \quad i = 1, \ldots, N. \tag{3.19}$$

By substituting (3.8), (3.17) and (3.18) into (3.19), it is seen that the overdetermined system has a solution only when $\mu_{\mathrm{G}}^i = \mu_{\mathrm{m}}$. In this case, the trial function $\tilde{w}$ becomes the actual warping displacement in the shaft and we recover the exact formula (Ting *et al.* 2004)

$$\mathcal{T}_N(\Omega) = \mathcal{A}^N. \tag{3.20}$$

It should be emphasized that the exact torsional rigidity (3.20) is independent of the shape Ω. In this case, the fibres are referred to as neutral inclusions (Chen *et al.* 2002), since the presence of the fibres do not disturb the warping field of the homogeneous shaft.

(*b*) *Reinforcements are cylindrically orthotropic coated fibres*

In this section we suppose that the reinforcements are cylindrically orthotropic fibres with cylindrically orthotropic coatings.

We solve the boundary-value problem within each coated fibre given by (3.12), (3.13) and (3.14) to obtain the explicit formula for trial warping function $\tilde{w}_i$ inside each fibre. From (3.12), it follows that

$$\tilde{w}_i^{\mathrm{f}} = \sum_{n=0}^{S} r^{n\lambda_{\mathrm{f}}}(A_n^{(\mathrm{f})} \cos n\theta + B_n^{(\mathrm{f})} \sin n\theta), \quad \text{for} \quad r \le a_i \quad \text{and}$$
$$\tilde{w}_i^{\mathrm{c}} = \sum_{n=0}^{S} r^{n\lambda_{\mathrm{c}}}(A_n^{(\mathrm{c})} \cos n\theta + B_n^{(\mathrm{c})} \sin n\theta) + \sum_{n=1}^{S} r^{-n\lambda_{\mathrm{c}}}\left(A_{-n}^{(\mathrm{c})} \cos n\theta + B_{-n}^{(\mathrm{c})} \sin n\theta\right),$$
$$\text{for } a_i \le r \le b_i, \tag{3.21}$$

where the coefficients $A_n^{(\mathrm{f})}$, $B_n^{(\mathrm{f})}$, $A_n^{(\mathrm{c})}$, $B_n^{(\mathrm{c})}$, $A_{-n}^{(\mathrm{c})}$ and $B_{-n}^{(\mathrm{c})}$ are unknown constants to be determined. Now the continuity conditions (3.13) at $r = a_i$ and boundary conditions (3.14) at $r = b_i$ require that

$$
\begin{aligned}
a_i^{n\lambda_\mathrm{f}} A_n^{(\mathrm{f})} &= a_i^{n\lambda_\mathrm{c}} A_n^{(\mathrm{c})} + a_i^{-n\lambda_\mathrm{c}} A_{-n}^{(\mathrm{c})}, \\
b_i^{n\lambda_\mathrm{c}} A_n^{(\mathrm{c})} + b_i^{-n\lambda_\mathrm{c}} A_n^{(\mathrm{c})} &= b_i^n \alpha_n, \\
\mu_\mathrm{G}^{i(\mathrm{f})} a_i^{n\lambda_\mathrm{f}} A_n^{(\mathrm{f})} &= \mu_\mathrm{G}^{i(\mathrm{c})} \left(a_i^{n\lambda_\mathrm{c}} A_n^{(\mathrm{c})} - a_i^{-n\lambda_\mathrm{c}} A_n^{(\mathrm{c})} \right) \quad \text{and}
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
a_i^{n\lambda_\mathrm{f}} B_n^{(\mathrm{f})} &= a_i^{n\lambda_\mathrm{c}} B_n^{(\mathrm{c})} + a_i^{-n\lambda_\mathrm{c}} B_{-n}^{(\mathrm{c})}, \\
b_i^{n\lambda_\mathrm{c}} B_n^{(\mathrm{c})} + b_i^{-n\lambda_\mathrm{c}} B_n^{(\mathrm{c})} &= b_i^n \beta_n, \\
\mu_\mathrm{G}^{i(\mathrm{f})} a_i^{n\lambda_\mathrm{f}} B_n^{(\mathrm{f})} &= \mu_\mathrm{G}^{i(\mathrm{c})} \left(a_i^{n\lambda_\mathrm{c}} B_n^{(\mathrm{c})} - a_i^{-n\lambda_\mathrm{c}} B_n^{(\mathrm{c})} \right).
\end{aligned}
\tag{3.23}
$$

The two linear systems (3.22) and (3.23) are entirely similar by observing that $A_n^{(\mathrm{f})} \leftrightarrow B_n^{(\mathrm{f})}$, $A_n^{(\mathrm{c})} \leftrightarrow B_n^{(\mathrm{c})}$, $A_{-n}^{(\mathrm{c})} \leftrightarrow B_{-n}^{(\mathrm{c})}$ and $\alpha_n \leftrightarrow \beta_n$. The solutions of (3.22) and (3.23) are derived as

$$
\begin{aligned}
\frac{A_n^{(\mathrm{f})}}{\alpha_n} &= \frac{B_n^{(\mathrm{f})}}{\beta_n} = \frac{2\nu_i^{(-n(1-\lambda_\mathrm{c}))/2} a_i^{n(1-\lambda_\mathrm{f})}}{(g^i+1) - \nu_i^{n\lambda_\mathrm{c}}(g^i-1)}, \\
\frac{A_n^{(\mathrm{c})}}{\alpha_n} &= \frac{B_n^{(\mathrm{c})}}{\beta_n} = \frac{(g^i+1)\nu_i^{(-n(1-\lambda_\mathrm{c}))/2} a_i^{n(1-\lambda_\mathrm{c})}}{(g^i+1) - \nu_i^{n\lambda_\mathrm{c}}(g^i-1)} \quad \text{and} \\
\frac{A_{-n}^{(\mathrm{c})}}{\alpha_n} &= \frac{B_{-n}^{(\mathrm{c})}}{\beta_n} = \frac{(1-g^i)\nu_i^{(-n(1-\lambda_\mathrm{c}))/2} a_i^{n(1-\lambda_\mathrm{c})}}{(g^i+1) - \nu_i^{n\lambda_\mathrm{c}}(g^i-1)}.
\end{aligned}
\tag{3.24}
$$

Substitution of $w_m|_{O_i}$ (see (3.8)) and the explicit formula for $\tilde{w}_i$ into (3.15) and (3.6) delivers the upper bound given in (2.8).

If the traction continuity (3.4) on the coating–matrix interface is also fulfilled, one has the additional constraint given by

$$
\mu_r^{i(\mathrm{c})} \frac{\partial \tilde{w}_i^\mathrm{c}}{\partial r}\bigg|_{r=b_i} = \mu_\mathrm{m} \frac{\partial w_\mathrm{m}|_{O_i}}{\partial r}\bigg|_{r=b_i}. \tag{3.25}
$$

Making use of (3.8) and (3.21), (3.25) can be rewritten as

$$
\begin{aligned}
\mu_\mathrm{m} \sum_{n=1}^{S} b_i^n \alpha_n &= \sum_{i=1}^{N} \mu_\mathrm{G}^{i(\mathrm{c})} \left(b_i^{n\lambda_\mathrm{c}} A_n^{(\mathrm{c})} - b^{-n\lambda_\mathrm{c}} A_{-n}^{(\mathrm{c})} \right) \quad \text{and} \\
\mu_\mathrm{m} \sum_{n=1}^{S} b_i^n \beta_n &= \sum_{i=1}^{N} \mu_\mathrm{G}^{i(\mathrm{c})} \left(b_i^{n\lambda_\mathrm{c}} B_n^{(\mathrm{c})} - b^{-n\lambda_\mathrm{c}} B_{-n}^{(\mathrm{c})} \right).
\end{aligned}
\tag{3.26}
$$

By substituting (3.24) into (3.26), it can be shown that the overdetermined system has a solution only when $\mu_\mathrm{m} = \mu_\mathrm{CCA}^i(n)$ for $i = 1, \ldots, N$ and $n = 1, \ldots, S$. In the latter case, the trial function $\tilde{w}$ becomes the actual warping displacement in the shaft and we recover the exact formula (Ting *et al.* 2004)

$$
\mathcal{T}_N(\Omega) = \mathcal{B}^N. \tag{3.27}
$$

4. Lower bounds on the torsional rigidity for shafts of arbitrary cross-section reinforced with coated or uncoated fibres

In this section we develop trial stress potentials for composite shafts with coated or uncoated fibres. These are substituted into the variational principle (2.3) to obtain the lower bounds given in (2.4) and (2.8). The trial potential to be constructed is denoted by $\tilde{\varphi}$. As in §3, we start by introducing the stress potential φ_{m} for a cross-section Ω filled with pure matrix material. In the matrix phase external to the coated or uncoated fibres we take the trial stress potential to be equal to φ_{m}. For future reference, we denote the restriction of the trial potential $\tilde{\varphi}$ to the matrix phase by $\tilde{\varphi}_{\mathrm{m}}$. In what follows it is convenient to represent φ_{m} in terms of local polar coordinates based at the centre O_i of the ith fibre. To proceed, we recall the relationship between the stress potential φ_{m} and the warping displacement w_{m} (e.g. Sokolnikoff 1956). Denoting the harmonic function conjugates to w_{m} by $\tilde{w}$, the stress potential φ_{m} is related to the warping displacement through

$$\varphi_{\mathrm{m}} = \mu_{\mathrm{m}}\left(\breve{w} - \frac{1}{2}|\boldsymbol{x}|^2\right). \tag{4.1}$$

We apply (4.1) together with the representation for $w_{\mathrm{m}}|_{O_i}$ given by (3.8) and the Cauchy Riemann equations in polar form to find that

$$\begin{aligned}\varphi_{\mathrm{m}}|_{O_i} &= \varphi_{\mathrm{m}}(r\cos\theta + \hat{x}_i, r\sin\theta + \hat{y}_i)\\ &= \mu_{\mathrm{m}}\left(\sum_{n=0}^{S} r^n(\alpha_n \sin n\theta - \beta_n \cos n\theta) - \frac{1}{2}r^2\right).\end{aligned} \tag{4.2}$$

To fix ideas, we note that for a circular cross-section with radius R, we find that the non-zero coefficients of α_n and β_n are given in (3.9) together with

$$\beta_0 = -\frac{1}{2}\left(R^2 - \hat{x}_i^2 - \hat{y}_i^2\right) = -\frac{1}{2}(R^2 - |\boldsymbol{x}_i|^2). \tag{4.3}$$

This agrees with previous known results (e.g. Lipton & Chen 2004).

(*a*) *Reinforcements are cylindrically orthotropic fibres*

We suppose that the cross-section Ω contains N cylindrically orthotropic fibres. The restriction of the trial field $\tilde{\varphi}$ to the interior of the ith fibre is denoted by $\tilde{\varphi}_i$. We choose $\tilde{\varphi}$ to be the superposition of two functions, i.e.

$$\tilde{\varphi}_i = \psi_i + \gamma_i, \tag{4.4}$$

where ψ^i satisfies

$$\frac{\partial^2\psi_i}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_i}{\partial r} + \frac{\lambda_i^2}{r^2}\frac{\partial^2\psi_i}{\partial\theta^2} = -2\mu_\theta^i, \quad \text{for } |\boldsymbol{x} - \boldsymbol{x}_i| = r < a_i, \tag{4.5}$$

with

$$\psi_i = 0, \quad \text{on } r = a_i \tag{4.6}$$

and the function γ^i satisfies

$$\frac{\partial^2 \gamma_i}{\partial r^2} + \frac{1}{r}\frac{\partial \gamma_i}{\partial r} + \frac{\lambda_i^2}{r^2}\frac{\partial^2 \gamma_i}{\partial \theta^2} = 0, \quad \text{for } r < a_i, \tag{4.7}$$

with

$$\gamma_i = \varphi_{\mathrm{m}}|_{O_i}, \quad \text{on} \quad r = a_i. \tag{4.8}$$

It should be noted that the function ψ_i is exactly the stress potential for a single cylindrically orthotropic fibre under torsion. The solution of the governing system (4.5) and (4.6) can be derived as

$$\psi_i = -\frac{\mu_\theta^i}{2}\left(r^2 - a_i^2\right). \tag{4.9}$$

The solution of γ^i is given by

$$\gamma_i = \sum_{n=0}^{N} r^{n\lambda}\left(C_n^i \cos m\theta + D_n^i \sin m\theta\right), \tag{4.10}$$

which, in reference to (4.8) and (4.2), will give

$$C_n^i = -\mu_{\mathrm{m}} a_i^{n(1-\lambda)}\beta_n, \quad D_n^i = \mu_{\mathrm{m}} a_i^{n(1-\lambda)}\alpha_n \quad \text{and} \quad C_0^i = -\mu_{\mathrm{m}}\left(\beta_0 + \frac{a_i^2}{2}\right). \tag{4.11}$$

The lower bound in (2.4) follows from substitution of this trial stress potential into the variational principle (2.3).

Suppose now that the trial potential field $\tilde{\varphi}$ satisfies the displacement continuity condition on the fibre–matrix interface given by

$$\mu_{\mathrm{m}}^{-1}\frac{\partial \tilde{\varphi}_{\mathrm{m}}}{\partial r}\bigg|_{r=a_i} = \left(\mu_\theta^i\right)^{-1}\frac{\partial \tilde{\varphi}_i}{\partial r}\bigg|_{r=a_i}. \tag{4.12}$$

This means that

$$\begin{aligned} -a_i + \sum_{n=1}^{\infty}\left(\mu_\theta^i\right)^{-1} n\lambda a_i^{n\lambda-1}\left(C_n^i \cos n\theta + D_n^i \sin n\theta\right) \\ = -a_i + \sum_{n=1}^{\infty} n a_i^{n-1}(-\beta_n \cos n\theta + \alpha_n \sin n\theta). \end{aligned} \tag{4.13}$$

It now follows from the formulae (4.11) for C_n^i and D_n^i that the condition (4.13) is satisfied only if $\mu_{\mathrm{G}}^i = \mu_{\mathrm{m}}$. For this situation, the trial $\tilde{\varphi}$ is the stress potential in the shaft and the exact formula (3.20) is recovered.

(*b*) *Reinforcements are cylindrically orthotropic coated fibres*

Here we derive the trial stress potential when the reinforcements are coated fibres. As in §4*a*, we choose the trial stress potential in the matrix $\tilde{\varphi}_{\mathrm{m}}$ to be φ_{m}. Inside each coated fibre, we take the trial function $\tilde{\varphi}_i$ to be given by the sum of two functions ψ_i and γ_i. Here $\psi_i^{(k)}$, k=f, c, satisfies

$$\frac{\partial^2 \psi_i^{(\mathrm{f})}}{\partial r^2}+\frac{1}{r}\frac{\partial \psi_i^{(\mathrm{f})}}{\partial r}+\frac{\lambda_i^{(\mathrm{f})2}}{r^2}\frac{\partial^2 \psi_i^{(\mathrm{f})}}{\partial \theta^2}=-2\mu_\theta^{i(\mathrm{f})} \quad \text{in the fibre,} \qquad r<a_i \quad \text{and} \tag{4.14}$$

$$\frac{\partial^2 \psi_i^{(\mathrm{c})}}{\partial r^2}+\frac{1}{r}\frac{\partial \psi_i^{(\mathrm{c})}}{\partial r}+\frac{\lambda_i^{(\mathrm{c})2}}{r^2}\frac{\partial^2 \psi_i^{(\mathrm{c})}}{\partial \theta^2}=-2\mu_\theta^{i(\mathrm{c})} \quad \text{in the coating,} \quad a_i<r<b_i. \tag{4.15}$$

At the fibre–coating interface and the coating–matrix interface, we require

$$\left[\psi_i^{(k)}\right]_{r=a_i}=0, \qquad \left[\frac{1}{\mu_\theta^{i(k)}}\frac{\partial \psi_i^{(k)}}{\partial r}\right]_{r=a_i}=0 \quad \text{and} \quad \psi_i^{(\mathrm{c})}\Big|_{r=b_i}=0. \tag{4.16}$$

The function γ^i satisfies

$$\begin{aligned}\frac{\partial^2 \gamma_i^{(\mathrm{f})}}{\partial r^2}+\frac{1}{r}\frac{\partial \gamma_i^{(\mathrm{f})}}{\partial r}+\frac{\lambda_i^{(\mathrm{f})2}}{r^2}\frac{\partial^2 \gamma_i^{(\mathrm{f})}}{\partial \theta^2}&=0, \quad \text{for} \quad r<a_i,\\ \frac{\partial^2 \gamma_i^{(\mathrm{c})}}{\partial r^2}+\frac{1}{r}\frac{\partial \gamma_i^{(\mathrm{c})}}{\partial r}+\frac{\lambda_i^{(\mathrm{c})2}}{r^2}\frac{\partial^2 \gamma_i^{(\mathrm{c})}}{\partial \theta^2}&=0, \quad \text{for } a_i<r<b_i \quad \text{and}\end{aligned} \tag{4.17}$$

$$\left[\gamma_i^{(k)}\right]_{r=a_i}=0, \qquad \left[\frac{1}{\mu_\theta^{i(k)}}\frac{\partial \gamma_i^{(k)}}{\partial r}\right]_{r=a_i}=0 \quad \text{and} \quad \gamma_i^{(\mathrm{c})}\big|_{r=b_i}=\tilde{\varphi}_{\mathrm{m}}\big|_{O_i}\big|_{r=b_i}. \tag{4.18}$$

Again, the function ψ_i is exactly the stress potential for a single coated fibre under torsion. The solution of the governing system (4.14)–(4.16) is given by

$$\begin{aligned}\psi_i^{(\mathrm{f})}&=-\frac{1}{2}\left(\mu_\theta^{i(\mathrm{f})}r^2-\mu_\theta^{i(\mathrm{c})}(b_i^2-a_i^2)-\mu_\theta^{i(\mathrm{f})}a_i^2\right)\\ \psi_i^{(\mathrm{c})}&=-\frac{1}{2}\mu_\theta^{i(\mathrm{c})}(r^2-b_i^2).\end{aligned} \tag{4.19}$$

The solutions $\gamma_i^{(k)}$, (4.17), are of the form

$$\begin{aligned}\gamma_i^{(\mathrm{f})}&=\sum_{n=0}^{S} r^{n\lambda_{\mathrm{f}}}\left(C_n^{(\mathrm{f})}\cos n\theta+D_n^{(\mathrm{f})}\sin n\theta\right), \quad \text{for } r\le a_i\\ \gamma_i^{(\mathrm{c})}&=\sum_{n=0}^{S} r^{n\lambda_{\mathrm{c}}}\left(C_n^{(\mathrm{c})}\cos n\theta+D_n^{(\mathrm{c})}\sin n\theta\right)+\sum_{n=1}^{S} r^{-n\lambda_{\mathrm{c}}}\left(C_{-n}^{(\mathrm{c})}\cos n\theta+D_{-n}^{(\mathrm{c})}\sin n\theta\right),\\ &\qquad \text{for } a_i\le r\le b_i\end{aligned} \tag{4.20}$$

where the coefficients $C_n^{(\mathrm{f})}$, $D_n^{(\mathrm{f})}$, $C_n^{(\mathrm{c})}$, $D_n^{(\mathrm{c})}$, $C_{-n}^{(\mathrm{c})}$ and $D_{-n}^{(\mathrm{c})}$ are unknown constants to be determined. Now the continuity conditions at $r=a_i$ and boundary conditions at $r = b_i$ give the relations

$$C_0^{(\mathrm{f})} = C_0^{(\mathrm{c})} = -\mu_\mathrm{m}\left(\beta_0 + \frac{b_i^2}{2}\right) \quad \text{and} \tag{4.21}$$

$$\begin{aligned} a_i^{n\lambda_\mathrm{f}} C_n^{(\mathrm{f})} &= a_i^{n\lambda_\mathrm{c}} C_n^{(\mathrm{c})} + a_i^{-n\lambda_\mathrm{c}} C_{-n}^{(\mathrm{c})}, \\ b_i^{n\lambda_\mathrm{c}} C_n^{(\mathrm{c})} + b_i^{-n\lambda_\mathrm{c}} C_{-n}^{(\mathrm{c})} &= -\mu_\mathrm{m} b_i^n \beta_n, \\ \left(\mu_\mathrm{G}^{i(\mathrm{f})}\right)^{-1} a_i^{n\lambda_\mathrm{f}} C_n^{(\mathrm{f})} &= \left(\mu_\mathrm{G}^{i(\mathrm{c})}\right)^{-1} \left(a_i^{n\lambda_\mathrm{c}} C_n^{(\mathrm{c})} - a_i^{-n\lambda_\mathrm{c}} C_{-n}^{(\mathrm{c})}\right) \quad \text{and} \end{aligned} \tag{4.22}$$

$$\begin{aligned} a_i^{n\lambda_\mathrm{f}} D_n^{(\mathrm{f})} &= a_i^{n\lambda_\mathrm{c}} D_n^{(\mathrm{c})} + a_i^{-n\lambda_\mathrm{c}} D_{-n}^{(\mathrm{c})}, \\ b_i^{n\lambda_\mathrm{c}} D_n^{(\mathrm{c})} + b_i^{-n\lambda_\mathrm{c}} D_{-n}^{(\mathrm{c})} &= \mu_\mathrm{m} b_i^n \alpha_n, \\ \left(\mu_\mathrm{G}^{i(\mathrm{f})}\right)^{-1} a_i^{n\lambda_\mathrm{f}} D_n^{(\mathrm{f})} &= \left(\mu_\mathrm{G}^{i(\mathrm{c})}\right)^{-1} \left(a_i^{n\lambda_\mathrm{c}} D_n^{(\mathrm{c})} - a_i^{-n\lambda_\mathrm{c}} D_{-n}^{(\mathrm{c})}\right), \end{aligned} \tag{4.23}$$

for each $n = 1, \ldots, S$. Again, the two linear systems (4.22) and (4.23) are entirely similar by observing that $C_n^{(\mathrm{f})} \leftrightarrow D_n^{(\mathrm{f})}$, $C_n^{(\mathrm{c})} \leftrightarrow D_n^{(\mathrm{c})}$, $C_{-n}^{(\mathrm{c})} \leftrightarrow D_{-n}^{(\mathrm{c})}$ and $\alpha_n \leftrightarrow -\beta_n$. The solutions of (4.22) and (4.23) are derived as

$$\begin{aligned} \frac{C_n^{(\mathrm{f})}}{\beta_n} &= -\frac{D_n^{(\mathrm{f})}}{\alpha_n} = -\frac{2\mu_\mathrm{m} g^i \nu_i^{(-n(1-\lambda_\mathrm{c}))/2} a_i^{n(1-\lambda_\mathrm{f})}}{(g^i+1) + \nu_i^{n\lambda_\mathrm{c}}(g^i-1)}, \\ \frac{C_n^{(\mathrm{c})}}{\beta_n} &= -\frac{D_n^{(\mathrm{c})}}{\alpha_n} = -\frac{\mu_\mathrm{m}(g^i+1)\nu_i^{(-n(1-\lambda_\mathrm{c}))/2} a_i^{n(1-\lambda_\mathrm{c})}}{(g^i+1) + \nu_i^{n\lambda_\mathrm{c}}(g^i-1)} \quad \text{and} \\ \frac{C_{-n}^{(\mathrm{c})}}{\beta_n} &= -\frac{D_{-n}^{(\mathrm{c})}}{\alpha_n} = -\frac{\mu_\mathrm{m}(1-g^i)\nu_i^{(-n(1-\lambda_\mathrm{c}))/2} a_i^{n(1+\lambda_\mathrm{c})}}{(g^i+1) + \nu_i^{n\lambda_\mathrm{c}}(g^i-1)}. \end{aligned} \tag{4.24}$$

Finally, we substitute of (4.1), (4.2), (4.19), (4.20) and (4.24) into (2.3), and after some tedious algebra, we obtain the lower bound given in (2.8).

Next suppose that the trial potential also satisfies the displacement continuity condition on the matrix–coating interface given by

$$\mu_\mathrm{m}^{-1} \frac{\partial \tilde{\varphi}_\mathrm{m}}{\partial r}\bigg|_{r=b_i} = \left(\mu_\theta^i\right)^{-1} \frac{\partial \tilde{\varphi}_i}{\partial r}\bigg|_{r=b_i}. \tag{4.25}$$

It follows from (4.2), (4.20), (4.24) and (3.24) that the additional requirement (4.25) gives an overdetermined system for the solution of the coefficients $C_n^{(\mathrm{f})}$, $D_n^{(\mathrm{f})}$, $C_n^{(\mathrm{c})}$, $D_n^{(\mathrm{c})}$, $C_{-n}^{(\mathrm{c})}$ and $D_{-n}^{(\mathrm{c})}$. This system has a solution only when $\mu_\mathrm{m} = \mu_\mathrm{CCA}^i(n)$ for $i = 1, \ldots, N$ and $n = 1, \ldots, S$. In this case, the trial function $\tilde{\varphi}$ becomes the stress potential inside the shaft and we recover the exact formula (Ting *et al.* 2004)

$$\mathcal{T}_N(\Omega) = \mathcal{B}^N. \tag{4.26}$$

5. Some examples

In this section we will derive results for a few simple cross-sectional shapes. Among various cross-sectional shapes of the composite shaft, circular cross-section is probably the most common one. Here, in addition to circular shaft, we will derive results for an elliptical Ω and an equilateral triangular Ω. In addition, simplified results will be given for shafts containing isotropic fibres or isotropic-coated fibres.

First, we recall that for the three cross-sectional shapes considered, the torsional rigidity $\mathcal{T}^{\mathrm{m}}(\Omega)$ for a homogenous Ω has exact expressions (Sokolnikoff 1956)

$$\begin{aligned} \mathcal{T}^{\mathrm{m}}(\Omega) &= \frac{\pi\mu_{\mathrm{m}}}{2}R^4, \quad \text{when } \Omega \text{ is a circle of radius } R, \\ \mathcal{T}^{\mathrm{m}}(\Omega) &= \pi\mu_{\mathrm{m}}\frac{A^3B^3}{A^2+B^2}, \quad \text{when } \Omega \text{ is an ellipse with semi-axes } A \text{ and } B, \\ \mathcal{T}^{\mathrm{m}}(\Omega) &= \frac{81\sqrt{3}}{80}\mu_{\mathrm{m}}l^4, \quad \text{when } \Omega \text{ is an equilateral triangle with side length } l. \end{aligned} \tag{5.1}$$

Next, using (3.9)–(3.11) into (2.6), we find that

$$\Xi^i = |\hat{\boldsymbol{x}}_i|^2 a_i^2, \tag{5.2}$$

for a circular Ω,

$$\Xi^i = \left[(\kappa^2+1)|\hat{\boldsymbol{x}}_i|^2 + 2\kappa\left(\hat{y}_i^2 - \hat{x}_i^2\right)\right]a_i^2 + \frac{1}{2}\kappa^2 a_i^4, \tag{5.3}$$

for an elliptical Ω, and

$$\Xi^i = a_i^2\left[\frac{3}{l^2}|\hat{\boldsymbol{x}}_i|^4 + \left(1+\frac{6a_i^2}{l^2}\right)|\hat{\boldsymbol{x}}_i|^2 + \frac{2\sqrt{3}\hat{x}_i\left(\hat{x}_i^2-\hat{y}_i^2\right)}{l} + \frac{a_i^4}{l^2}\right], \tag{5.4}$$

for an equilateral triangular Ω with side length l. The term for $\sum_{n=1}^{S} \Upsilon^i(n)$ is similar to that of Ξ^i except by replacing a_i with b_i. Also, the term $\sum_{n=1}^{S} \Upsilon^i(n)\mu^i_{\mathrm{CCA}}(n)$ in (2.8) can be expanded in the forms

$$\sum_{n=1}^{S} \Upsilon^i(n)\mu^i_{\mathrm{CCA}}(n) = |\hat{\boldsymbol{x}}_i|^2 b_i^2 \mu^i_{\mathrm{CCA}}(1), \tag{5.5}$$

for a circular Ω,

$$\sum_{n=1}^{S} \Upsilon^i(n)\mu^i_{\mathrm{CCA}}(n) = \left[(\kappa^2+1)|\hat{\boldsymbol{x}}_i|^2 + 2\kappa\left(\hat{y}_i^2-\hat{x}_i^2\right)\right]\mu^i_{\mathrm{CCA}}(1)b_i^2 + \frac{1}{2}\kappa^2\mu^i_{\mathrm{CCA}}(2)b_i^4, \tag{5.6}$$

for an elliptical Ω, and

$$\begin{aligned} \sum_{n=1}^{S} \Upsilon^i(n)\mu^i_{\mathrm{CCA}}(n) = b_i^2\Bigg[&\left(\frac{3}{l^2}|\hat{\boldsymbol{x}}_i|^4 + |\hat{\boldsymbol{x}}_i|^2 + \frac{2\sqrt{3}\hat{x}_i\left(\hat{x}_i^2-\hat{y}_i^2\right)}{l}\right)\mu^i_{\mathrm{CCA}}(1) \\ &+ \frac{6b_i^2}{l^2}|\hat{\boldsymbol{x}}_i|^2\mu^i_{\mathrm{CCA}}(2) + \frac{b_i^4}{l^2}\mu^i_{\mathrm{CCA}}(3)\Bigg], \end{aligned} \tag{5.7}$$

for an equilateral triangle. Upon substitution of (5.1)–(5.7) into (2.4) and (2.8), we can find torsional rigidity bounds for each case. Here explicit results will only be stated for circular shafts. When the shaft has circular cross-sections with radius R, we denote Ω as D_R. The torsional rigidity of D_R reinforced with N fibres is written as $\mathcal{T}_N(D_R)$.

Proposition 5.1. *For a shaft with circular cross-section containing N cylindrically orthotropic fibres, the bounds for the torsional rigidity are*

$$\mathcal{A}^N + \pi\sum_{i=1}^{N}|\hat{\boldsymbol{x}}_i|^2 a_i^2 \frac{\mu_{\mathrm{m}}}{\mu_{\mathrm{G}}^i}\left(\mu_{\mathrm{G}}^i - \mu_{\mathrm{m}}\right) \le \mathcal{T}_N(D_R) \le \mathcal{A}^N + \pi\sum_{i=1}^{N}|\hat{\boldsymbol{x}}_i|^2 a_i^2\left(\mu_{\mathrm{G}}^i - \mu_{\mathrm{m}}\right), \tag{5.8}$$

where $\mathcal{A}^N$ is now given by

$$\mathcal{A}^N = \frac{\pi}{2}\mu_{\mathrm{m}}R^4 + \frac{\pi}{2}\sum_{i=1}^{N}\left(\mu_\theta^i - \mu_{\mathrm{m}}\right)a_i^4. \tag{5.9}$$

Proposition 5.2. *For a circular shaft reinforced with N cylindrically orthotropic coated fibres, the bounds for the torsional rigidity are*

$$\begin{aligned}\mathcal{B}^N + \pi\sum_{i=1}^{N}|\hat{\boldsymbol{x}}_i|^2 b_i^2 \frac{\mu_{\mathrm{m}}}{\mu_{\mathrm{CCA}}^i}\left(\mu_{\mathrm{CCA}}^i(1) - \mu_{\mathrm{m}}\right) &\le \mathcal{T}_N(D_R)\\ &\le \mathcal{B}^N + \pi\sum_{i=1}^{N}|\hat{\boldsymbol{x}}_i|^2 b_i^2\left(\mu_{\mathrm{CCA}}^i(1) - \mu_{\mathrm{m}}\right),\end{aligned} \tag{5.10}$$

where $\mathcal{B}^N$ is given by

$$\mathcal{B}^N = \frac{\pi}{2}\mu_{\mathrm{m}}R^4 + \frac{\pi}{2}\sum_{i=1}^{N}\left(\mu_\theta^{i(\mathrm{c})}\left(b_i^4 - a_i^4\right) + \mu_\theta^{i(\mathrm{f})}a_i^4 - \mu_{\mathrm{m}}b_i^4\right). \tag{5.11}$$

When the constituents are isotropic, then $\mu_r^{i(k)} = \mu_\theta^{i(k)}$, k=f, c. It is also seen from (3.12) that $\lambda_k = 1$ and $g^i = \mu^{i(\mathrm{f})}/\mu^{i(\mathrm{c})}$. The bounds in (5.10) recover our previous results (proposition 2.1; Lipton & Chen 2004). While the expression (5.8) constitutes bounds for shafts containing homogeneous cylinders with different materials. These latter upper and lower bounds will coincide only when the shaft is homogeneous, $\mu_{\mathrm{m}} = \mu_{\mathrm{f}}$.

6. Generalization to multicoated fibres

We have seen in (2.4) that the bounds will reduce to the exact result $\mathcal{A}^N$ when $\mu_{\mathrm{m}} = \mu_{\mathrm{G}}^i$. This finding, first reported by Ting *et al.* (2004), is generally true regardless of the geometric shape of Ω. Mathematically, this means that a host medium with anti-plane shear rigidity μ_{G} is neutral to a cylindrically orthotropic fibre (with shear rigidities μ_r and μ_θ) under *various orders* of boundary fields. However, when the inclusion is a coated fibre, made up of two constituent materials, the condition of $\mu_{\mathrm{m}} = \mu_{\mathrm{CCA}}^i(n)$ may not be easily fulfilled for each

$n=1,\dots,S$ (Chen 2004). For example, in (3.10) we have demonstrated that an elliptical Ω corresponds $n=1$ and 2. Chen (2004) has demonstrated that for a coated fibre it is not possible to find a non-trivial set of phase constituents that satisfy the condition $\mu^i_{\mathrm{CCA}}(1)=\mu^i_{\mathrm{CCA}}(2)$. This means that, for an elliptical Ω, the upper and lower bounds given in (2.8) will not coincide. In order to construct a class of composite shafts for which the bounds recover exact results, we may consider the fibres to be multicoated or even graded, as suggested by Chen (2004). These multiphase fibres have more degrees of freedom to satisfy the neutrality conditions.

To derive bounds for the torsional rigidity for an arbitrary shaft containing N multicoated fibres, one still proceeds as in §§3*b* and 4*b* for coated fibres. Let us consider a circular multicoated fibre Σ, with outer radius a^i_1, $i=1,\dots,N$. The multicoated cylinder consists of a core with radius a_Q and $(Q-1)$ layers of coating. The qth layer of the coating occupies the annulus $a_{q+1}\le r\le a_q$, $q=1,2,\dots,Q-1$, $a_{Q+1}=0$. The auxiliary boundary-value problems for the trial warping field and trial stress potential, though more tedious to obtain, are entirely similar to those of coated fibres. Here without repeating the algebraic process, following the bounds obtained for the coated fibre, proposition 2.2, and the exact result found in Chen (2004) and Ting *et al.* (2004), we propose the following bounds for the torsional rigidity of a shaft containing N multicoated fibres.

Proposition 6.1. *For a shaft with arbitrary cross-section Ω containing N cylindrically orthotropic multicoated fibres, the torsional rigidity of the reinforced shaft $\mathcal{T}_N(\Omega)$ is bounded between the quantities*

$$\mathcal{C}^N+\pi\sum_{i=1}^{N}\sum_{n=1}^{S}\Theta^i(n)\frac{\mu_{\mathrm{m}}}{\mu^i_{\mathrm{MCA}}(n)}\left(\mu^i_{\mathrm{MCA}}(n)-\mu_{\mathrm{m}}\right)\le\mathcal{T}_N(\Omega)$$

$$\le\mathcal{C}^N+\pi\sum_{i=1}^{N}\sum_{n=1}^{S}\Theta^i(n)\left(\mu^i_{\mathrm{MCA}}(n)-\mu_{\mathrm{m}}\right), \tag{6.1}$$

where $\mathcal{C}^N$ and Θ^i are given by

$$\mathcal{C}^N=\mathcal{T}^{\mathrm{m}}(\Omega)+\frac{\pi}{2}\sum_{i=1}^{N}\sum_{q=1}^{Q}\left\{\mu_\theta^{i(q)}\left[\left(a^i_q\right)^4-\left(a^i_{q+1}\right)^4\right]-\mu_{\mathrm{m}}\left(a^i_1\right)^4\right\}, \tag{6.2}$$

$$\Theta^i(n)=n\left(\left(\alpha^i_n\right)^2+\left(\beta^i_n\right)^2\right)\left(a^i_1\right)^{2n} \tag{6.3}$$

and $\mu^i_{\mathrm{MCA}}(n)$ is the anti-plane shear rigidity of an assemblage of multicoated cylinders under boundary fields of order n.

The formula of $\mu^i_{\mathrm{MCA}}(n)$ has been derived in Ting *et al.* (2004) in construction of a neutral multicoated cylinder under torsion. For completeness, we record the expression in appendix A.

T.C. was supported by the National Science Council, Taiwan, under contract NSC 95-2211-E006-466. R.L. was supported by NSF grant DMS-0406374, AFOSR grant F49620-02-1-0041 and the Boeing Aircraft Company.

Appendix A

In this appendix, we listed the formula of the anti-plane shear rigidity of an assemblage of multicoated cylinders under boundary fields of order n. The detailed derivation steps can be found in the work of Ting *et al.* (2004). Here we only recorded the formula based on the notations defined in the main text

$$\mu^{i}_{\mathrm{MCA}}(n) = \frac{K^{(Q,n)}_{11} - K^{(Q,n)}_{21}}{K^{(Q,n)}_{22} - K^{(Q,n)}_{12}}\,\mu^{(1)}_{\mathrm{G}}, \tag{A 1}$$

where the index 'i' denotes the ith multicoated cylinder,

$$\boldsymbol{K}^{(Q,n)} = \boldsymbol{K}^{(Q-1)}_{n}\boldsymbol{K}^{(Q-2)}_{n}\ldots\boldsymbol{K}^{(1)}_{n}, \tag{A 2}$$

with

$$\boldsymbol{K}^{(q)}_{n} = \begin{pmatrix} c^{(q)}_{n} & -c^{(q)}_{-n} \\ -c^{(q)}_{-n}h_q & c^{(q)}_{n}h_q \end{pmatrix} \quad \text{and} \tag{A 3}$$

$$h_q = \mu^{(q)}_{\mathrm{G}}/\mu^{(q+1)}_{\mathrm{G}}, \qquad c^{(q)}_{\pm n} = (a_{q+1}/a_q)^{-n\lambda_q} \pm (a_{q+1}/a_q)^{n\lambda_q}. \tag{A 4}$$

We note that, for clarity and without adding many indices, we have omitted the index i on the r.h.s. of (A 1) and (A 2)–(A 4).

References

Alvino, A. & Trombetti, G. 1985 Isoperimetric inequalities connected with torsion problem and capacity. *Boll. Un. Mat. Ital. B* **4**, 773–787.

Chen, T. 2004 An exactly solvable microgeometry in torsion: assemblage of multicoated cylinders. *Proc. R. Soc. A* **460**, 1981–1993. (doi:10.1098/rspa.2003.1268)

Chen, T. & Wei, C.-J. 2005 Saint-Venant's torsion of anisotropic shafts: theoretical frameworks, extremal bounds and affine transformation. *Q. J. Mech. Appl. Math.* **58**, 269–287. (doi:10.1093/qjmamj/hbi013)

Chen, T., Benveniste, Y. & Chuang, P. C. 2002 Exact solutions in torsion of composite bars: thickly coated neutral inhomogeneities and composite cylinder assemblages. *Proc. R. Soc. A* **458**, 1719–1759. (doi:10.1098/rspa.2001.0933)

de Saint-Venant, B. 1856 Memoire sur la torsion des prismes. *Memor. pres. divers savants Acad. Sci.* **14**, 233–560.

Diaz, J. B. & Weinstein, A. 1948 The torsional rigidity and variational methods. *Am. J. Math.* **70**, 107–116. (doi:10.2307/2371935)

Lipton, R. 1998 Optimal fiber configurations for maximum torsional rigidity. *Arch. Ration. Mech. Anal.* **144**, 79–106. (doi:10.1007/s002050050113)

Lipton, R. 1999 An isoperimetric inequality for the torsional rigidity of imperfectly bonded fiber reinforced shafts. *J. Elast.* **55**, 1–10. (doi:10.1023/A:1007606330007)

Lipton, R. & Chen, T. 2004 Bounds and extremal configurations for the torsional rigidity of coated fiber reinforced shafts. *SIAM J. Appl. Math.* **65**, 299–315. (doi:10.1137/S0036139903424229)

Lipton, R. & Vernescu, B. 1995 Variational methods, size effects, and extremal microgeometries for elastic composites with imperfect interface. *Math. Methods Models Appl. Sci.* **5**, 1139–1173. (doi:10.1142/S0218202595000607)

Milton, G. W. 2002 *The theory of composites*. Cambridge, UK: Cambridge University Press.

Payne, L. E. 1962 Some isoperimetric inequalities in the torsion problem for multiply connected regions. In *Studies in mathematical analysis and related topics. Essay in honor of G. Polya*. Stanford, CA: Stanford University Press.

Payne, L. E. & Weinberger, H. F. 1961 Some isoperimetric inequalities for membrane frequencies and torsional rigidity. *J. Math. Anal. Appl.* **2**, 210–216. (doi:10.1016/0022-247X(61)90031-2)

Polya, G. 1948 Torsional rigidity, principal frequency, electrostatic capacity and symmetrization. *Q. Appl. Math.* **6**, 267–277.

Polya, G. & Weinstein, A. 1950 On the torsional rigidity of multiply connected cross sections. *Ann. Math.* **52**, 155–163.

Sokolnikoff, I. S. 1956 *Mathematical theory of elasticity*. New York, NY: McGraw-Hill.

Ting, T. C. T., Chen, T. & Li, K. S. 2004 Torsion of an isotropic shaft of arbitrary cross-section embedded with multicoated or graded circular cylinders of cylindrically orthotropic materials. *Q. J. Mech. Appl. Math.* **57**, 347–362. (doi:10.1093/qjmam/57.3.347)