

# Relaxation through Homogenization for Optimal Design Problems with Gradient Constraints<sup>1</sup>

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**Abstract.** The problem of the relaxation of optimal design problems for multiphase composite structures in the presence of constraints on the gradient of the state variable is addressed. A relaxed formulation for the problem is given in the presence of either a finite or infinite number of constraints. The relaxed formulation is used to identify minimizing sequences of configurations of phases.

**Key Words.** Gradient constraints, relaxation, optimal design.

## 1. Introduction

In the study of the optimal design of composite structures, much effort has focused on the problems of optimization of design criteria in the absence of constraints on the gradients of the field variables. In this article, a theory is developed for the relaxation of optimal design problems for multiphase composite structures in the presence of constraints on the gradient. An example of where such a theory would be useful is the design of a multiphase insulator with a prescribed overall dielectric constant subject to the constraint that, inside the composite, the electric field does not exceed a critical value. This type of problem is important in the applications (Ref. 1), where large values of the local electric field are attributed to material breakdown (Ref. 2).

To fix the ideas, the paper is written in the physical context of multiphase dielectric materials. The design domain is a bounded open set  $\Omega$  in  $R^m$  and the multiphase dielectric is made up of  $N$  anisotropic dielectric materials with dielectric constants given by the  $m \times m$  symmetric positive

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matrices  $A_1, A_2, A_3, \dots, A_N$ . A particular choice of the component dielectric constants is specified by the array

$$\underline{A} = (A_1, A_2, A_3, \dots, A_N)$$

and the dielectric constant at each point in the design domain is given by

$$C(\underline{A}, x) = \sum_{i=1}^N \chi_i(x) A_i.$$

Here,  $\chi_i$  is the indicator function of the set occupied by the  $i$ th dielectric phase, with  $\chi_i = 1$  for  $x$  in the  $i$ th phase and  $\chi_i = 0$  elsewhere. The design space consists of all partitions of  $\Omega$  into  $N$  Lebesgue measurable subsets  $\Omega_1, \Omega_2, \dots, \Omega_N$  occupied by the different dielectrics subject to the resource constraints

$$\text{meas}(\Omega_i) \leq \gamma_i.$$

Here,

$$\sum_i \gamma_i \geq \text{meas}(\Omega),$$

and the vector of resource constraints is written as

$$\underline{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_N).$$

The electric potential in the composite is denoted by  $\phi$  and, for a prescribed distributed charge density,  $f$  satisfies

$$-\text{div}(C(\underline{A}, x)\nabla\phi) = f. \quad (1)$$

Here, the potential  $\phi$  is zero on the boundary of  $\Omega$ . From the theory of the Poisson equation, it is well known that the potential lies in the Sobolev space  $H_0^1(\Omega)$  and  $f$  can be chosen from the dual space  $H^{-1}(\Omega)$ . The design criteria is a function of the potential and is denoted by  $F(\phi)$ . The types of criteria considered here are those associated with overall energy dissipation, given by

$$F(\phi) = \int_{\Omega} \phi f \, dx,$$

or the distance of the potential from a desired target potential  $\hat{\phi}$ , given by

$$F(\phi) = \int_{\Omega} |\phi - \hat{\phi}|^2 \, dx.$$

Both of these objective functions are continuous with respect to weak convergence in  $H_0^1(\Omega)$ . The results given here are not restricted to this situation

and apply to any design criteria that is continuous with respect to G-convergent sequences of designs (Ref. 3).

For a given tolerance  $K > 0$  and a prescribed nonnegative function  $p(x)$ , the basic gradient constraint is given by

$$\int_{\Omega} p(x) |\nabla \phi|^2 dx \leq K^2. \tag{2}$$

For simplicity,  $p(x)$  is chosen to be in the space of infinitely differentiable functions continuous on the closure of  $\Omega$ , i.e.,  $p(x)$  in  $C^\infty(\bar{\Omega})$ . For a finite collection of functions  $p_1, p_2, \dots, p_j$ , one has the set of constraints given by

$$\int_{\Omega} p_k(x) |\nabla \phi|^2 dx \leq K^2, \quad p_k(x) \geq 0, \quad k = 1, 2, \dots, j. \tag{3}$$

Last, one may consider the infinite number of constraints given by

$$\int_{\Omega} p(x) |\nabla \phi|^2 dx \leq K^2 \int_{\Omega} p(x) dx, \quad p(x) \geq 0 \text{ in } C^\infty(\bar{\Omega}), \tag{4}$$

which is equivalent to the local constraint

$$|\nabla \phi| \leq K, \quad \text{almost everywhere in } \Omega. \tag{5}$$

The basic gradient-constrained optimal design problem is given by

$$P = \inf_{\substack{\text{configurations,} \\ \text{meas}(\Omega_i) = \gamma_i}} F(\phi), \tag{6}$$

where  $\phi$  is a solution of the equation of state (1) and is subject to the gradient constraint given by (2). Optimal design problems subject to either a finite or an infinite number of gradient constraints are formulated in the same way.

The goal of this analysis is to present a theory of relaxation for these problems that provides the theoretical underpinnings for a numerical approach to the solution of the design problem. The problem as stated above is not readily amenable to numerical solution. The fundamental reason for this is that problems of this type do not possess configurations for which the infimum in (6) is obtained; see Refs. 4–6. Thus, any approach that seeks to identify optimal configurations is most likely to fail. Instead, one must seek a methodology for systematically identifying minimizing sequences of configurations that approach the infimum of (6).

In the absence of gradient constraints, problems of this kind have been the object of intense and fruitful study. One approach to these problems is the relaxation of the design problem through the extension of the design

space. This is accomplished using the theory of homogenization. This approach can be found in the seminal work of Ref. 7 and Ref. 8. The approach taken here follows the philosophy given in Ref. 7 and Ref. 8 and provides a relaxation of the design problem (6) through an extension of the design space. However, the extended design space for the gradient-constrained design problem is seen to be a generalization of the one developed for the unconstrained problem. Loosely speaking, in the absence of gradient constraints, the design space is extended to include all effective dielectric tensors made from composites formed from the constituent dielectric materials. However, for gradient-constrained optimal design problems, it is found that the relaxed problem requires the extension of the design space to include all the effective dielectric tensors viewed as functions of the dielectric tensors of the constituent materials.

In order to proceed, the optimal design problem over configurations is reformulated in an equivalent way. For a given array  $\underline{A}$ , a neighborhood  $\mathcal{N}(\underline{A})$  of  $\underline{A}$  is introduced. Arrays in this neighborhood are denoted by

$$\underline{P} = (P_1, P_2, \dots, P_N).$$

The neighborhood is chosen such that all matrices  $P_i$  in the array  $\underline{P}$  satisfy the constraint

$$0 < \lambda < P_i < \Lambda.$$

The associated set of controls

$$C(\underline{P}, x) = \sum_{i=1}^N \chi_i(x) P_i,$$

for  $\underline{P}$  in the neighborhood  $\mathcal{N}(\underline{A})$  for which  $\int_{\Omega} \chi_i dx \leq \gamma_i$ , is denoted by  $\mathcal{C}(\underline{A}, \underline{\gamma})$ , and the design problem is given by

$$P = \inf_{C(\underline{P}, x) \in \mathcal{C}(\underline{A}, \underline{\gamma})} F(\phi), \quad (7)$$

where  $\phi$  is a solution of the equation of state (1) and is subject to the gradient constraint given by (2). As before, optimal design problems subject to either a finite or an infinite number of gradient constraints are formulated in the same way.

The notion of homogenization is given in the theories of G-convergence and H-convergence; see Ref. 9 and Ref. 10. Both notions of convergence coincide for the problem addressed here. A comprehensive introduction to the theory of G-convergence together with extensive references to the literature is given in Ref. 11.

**Definition 1.1.** G-Convergence. The sequence  $\{C^n(\underline{P}, x)\}_{n=1}^\infty$  G-converges to  $C^\infty(\underline{P}, x)$  if and only if, for any open set  $\omega \subset \subset \Omega$  and any  $f$  in  $H^{-1}(\omega)$ , the solutions  $w^n$  in  $H_0^1(\omega)$  of

$$-\operatorname{div}(C^n(\underline{P}, x)\nabla w^n) = f \tag{8}$$

satisfy

$$\begin{aligned} w^n &\rightharpoonup w, && \text{weakly in } H_0^1(\omega), \\ C^n(\underline{P}, x)\nabla w^n &\rightharpoonup C^\infty(\underline{P}, x)\nabla w, && \text{weakly in } L^2(\omega)^N, \end{aligned}$$

where  $w$  is the  $H_0^1(\omega)$  solution of

$$-\operatorname{div}(C^\infty(\underline{P}, x)\nabla w) = f. \tag{9}$$

The resource constraints for a G-convergent sequence of controls is handled in the usual way by assuming that the associated sequence of characteristic functions  $\{\chi_i^n\}_{n=1}^\infty$  satisfy

$$\chi_i^n \rightharpoonup \theta_i, \quad \text{in } L^\infty(\Omega) \text{ weak}^*. \tag{10}$$

The design space is extended to include the set of all functions  $C^\infty(\underline{P}, x)$  for which there exists a sequence of controls  $\{C^n(\underline{P}, x)\}_{n=1}^\infty$  G-converging to  $C^\infty(\underline{P}, x)$  for all  $\underline{P}$  in  $\mathcal{N}(\underline{A})$  and  $\int_\Omega \theta_i dx \leq \gamma_i$ . The set of all such functions  $C^\infty(\underline{P}, x)$  is the G-closure of the set  $\mathcal{C}(\underline{A}, \gamma)$  and is denoted by  $\mathcal{GC}(\underline{A}, \gamma)$ . We point out that  $\mathcal{C}(\underline{A}, \gamma)$  is contained in  $\mathcal{GC}(\underline{A}, \gamma)$ . Its evident that the extension of the design space introduced here is more general than the usual one in that it involves the notion of tensor-valued effective dielectric functions as opposed to effective dielectric tensors.

The next step in the relaxation procedure is the correct description of the gradient constraints (2)–(5) in the extended design space. Given a sequence  $\{C^n(\underline{P}, x)\}_{n=1}^\infty$  that G-converges to  $C^\infty(\underline{P}, x)$  for every  $\underline{P}$  in  $\mathcal{N}(\underline{A})$ , the local corrector function associated with the sequence  $\{C^n(\underline{A}, x)\}_{n=1}^\infty$  is introduced. Consider a cube  $Q(x, r)$  of side length  $r$  centered at some point  $x$  in  $\Omega$ . For  $r$  sufficiently small,  $Q(x, r)$  is contained within  $\Omega$ . Given a vector  $E$  in  $R^N$ , the local corrector function is written as  $\varphi_E^{n,r}$ , where  $\varphi_E^{n,r}$  is the  $H_0^1(Q(x, r))$  solution of

$$-\operatorname{div}(C^n(\underline{A}, y)(\nabla \varphi_E^{n,r} + E)) = -\operatorname{div}(C^\infty(\underline{A}, y)E), \quad \text{for } y \text{ in } Q(x, r). \tag{11}$$

The local corrector functions are used to define the directional derivatives of the G-limit  $C^\infty(\underline{P}, x)$  in the following theorem.

**Theorem 1.1.** Given that  $\{C^n(\underline{P}, x)\}_{n=1}^\infty$  G-converges to  $C^\infty(\underline{P}, x)$  for every  $\underline{P}$  in  $\mathcal{N}(\underline{A})$ , the directional derivative of  $C^\infty(\underline{P}, x)$  at  $\underline{A}$  with respect

to the  $i$ th component conductivity in the direction specified by the  $m \times m$  symmetric matrix  $M_i$  is given by

$$\begin{aligned} & \partial C^\infty / \partial M_i E \cdot E \\ &= \sum_{k=1}^m \sum_{l=1}^m [M_i]_{kl} \left[ \lim_{r \rightarrow 0} \lim_{n \rightarrow \infty} (1/|Q(x, r)|) \right. \\ & \quad \left. \times \int_{Q(x, r)} \chi_i^n (\nabla_k \phi_E^{n, r} + E_k) (\nabla_l \phi_E^{n, r} + E_l) dy \right] \end{aligned} \quad (12)$$

and exists almost everywhere in  $\Omega$  and for every  $E$  in  $R^m$ .

From Theorem 1.1, it follows that the definition of the  $i$ th phase gradient, denoted by  $(\nabla_{kl}^i C^\infty(A, x))E \cdot E$ , is given by the local formula

$$\begin{aligned} & (\nabla_{kl}^i C^\infty(A, x))E \cdot E \\ &= \lim_{r \rightarrow 0} \lim_{n \rightarrow \infty} (1/|Q(x, r)|) \int_{Q(x, r)} \chi_i^n (\nabla_k \phi_E^{n, r} + E_k) (\nabla_l \phi_E^{n, r} + E_l) dy \end{aligned} \quad (13)$$

and

$$\begin{aligned} & (\nabla_{kk}^i C^\infty(A, x))E \cdot E \\ &= \lim_{r \rightarrow 0} \lim_{n \rightarrow \infty} (1/|Q(x, r)|) \int_{Q(x, r)} \chi_i^n |\nabla \phi_E^{n, r} + E|^2 dy, \end{aligned} \quad (14)$$

where repeated indices indicate summation. Theorem 1.1 provides explicit formulas for the derivatives of the G-limit in terms of the microscopic problems (11). The identification of the suitable microscopic problems together with the local formulas (12)–(14) are necessary for numerical solution schemes based on relaxation through homogenization; see Ref. 12. General differentiability properties of the G-limits obtained without the use of local formulas are presented in Ref. 13. Differentiability properties for the effective dielectric tensor in the contexts of periodic homogenization and statistically homogeneous random media were presented in Ref. 14.

The relaxation scheme developed here follows from the homogenization result given below. In order to state the result, one considers a sequence of configurations and a function  $C^\infty(P, x)$  for which the sequence of controls  $\{C^n(P, x)\}_{n=1}^\infty$  G-converges to  $C^\infty(P, x)$  for every  $P$  in  $\mathcal{S}(A)$ . From homogenization theory, there exists a potential  $\phi^\infty$  such that the sequence of potentials  $\{\phi^n\}_{n=1}^\infty$  in  $H_0^1(\Omega)$  satisfying

$$-\text{div}(C^n(A, x)\nabla\phi^n) = f \quad (15)$$

converges weakly to  $\phi^\infty$  in  $H_0^1(\Omega)$ , where

$$-\text{div}(C^\infty(\underline{A}, x)\nabla\phi^\infty) = f. \tag{16}$$

The following homogenization theorem allows one to pass to the homogenization limit in the gradient constraints of the form (2)–(5).

**Theorem 1.2.** Homogenization Limit of Weakly Converging Gradients. For any function  $p(x)$  in  $C^\infty(\bar{\Omega})$ ,

$$\lim_{n \rightarrow \infty} \int_{\Omega} p(x)|\nabla\phi^n|^2 dx = \int_{\Omega} p(x)(\sigma(x) + I)\nabla\phi^\infty \cdot \nabla\phi^\infty dx, \tag{17}$$

where the covariance  $\sigma(x)$  is a nonnegative matrix defined by

$$\sigma(x)E \cdot E = \sum_{i=1}^N (\nabla_{kk}^i C^\infty(\underline{A}, x))E \cdot E - |E|^2, \tag{18}$$

for all vectors  $E$  in  $R^m$ .

The homogenization limit of the gradient constraints is given in the following corollary.

**Corollary 1.1.** Homogenization Limit of Gradient Constraints. For some nonnegative  $p$  in  $C^\infty(\bar{\Omega})$ , if it is known that

$$\int_{\Omega} p(x)|\nabla\phi^n|^2 dx \leq K^2 \tag{19}$$

for the sequence  $\{\phi^n\}_{n=1}^\infty$  given above, then  $\nabla\phi^\infty$  satisfies

$$\int_{\Omega} p(x)(\sigma(x) + I)\nabla\phi^\infty \cdot \nabla\phi^\infty dx \leq K^2. \tag{20}$$

Similarly, if one has the local constraint  $|\nabla\phi^n| \leq K$  almost everywhere in  $\Omega$ , then  $\nabla\phi^\infty$  satisfies

$$\sqrt{(\sigma(x) + I)\nabla\phi^\infty \cdot \nabla\phi^\infty} \leq K, \tag{21}$$

almost everywhere in  $\Omega$ .

The covariance can be thought of as a new effective property characterizing the interaction between the local microstructure and the macroscopic electric field  $-\nabla\phi^\infty$ . It has been introduced in the contexts of periodic and statistically homogeneous random composites in Refs. 15–16.

In the general context of G-convergent sequences of matrices, the local formula for  $\sigma$  is given in Theorem 1.1 of Ref. 17.

The relaxed formulation for the basic gradient-constrained optimal design problem is introduced. For  $p(x) \geq 0$  and  $p(x)$  in  $C^\infty(\bar{\Omega})$ , the relaxed formulation of (7) is

$$QP = \inf_{C^\infty(P, x) \text{ in } \mathcal{C}(A, \gamma)} F(\phi), \tag{22}$$

subject to the constraint

$$\int_{\Omega} p(x)(\sigma(x) + I)\nabla\phi \cdot \nabla\phi \, dx \leq K^2, \tag{23}$$

where  $\phi$  is the  $H_0^1(\Omega)$  solution to the state equation

$$-\text{div}(C^\infty(A, x)\nabla\phi) = f, \tag{24}$$

and  $\sigma$  is defined by (18).

The first important feature of the relaxed problem is that the infimum of the relaxed design problem is attained by a control in  $\mathcal{C}(A, \gamma)$ .

**Theorem 1.3.** There exists a control  $\hat{C}^\infty(P, x)$  in  $\mathcal{C}(A, \gamma)$  and a state variable  $\hat{\phi}$  in  $H_0^1(\Omega)$  for which

$$-\text{div}(\hat{C}^\infty(A, x)\nabla\hat{\phi}) = f \tag{25}$$

and

$$QP = F(\hat{\phi}), \tag{26}$$

where

$$\int_{\Omega} p(x)(\hat{\sigma}(x) + I)\nabla\hat{\phi} \cdot \nabla\hat{\phi} \, dx \leq K^2 \tag{27}$$

and

$$\hat{\sigma}(x)E \cdot E = \sum_{i=1}^N (\nabla_{kk}^i \hat{C}^\infty(A, x))E \cdot E - |E|^2, \tag{28}$$

for every  $E$  in  $R^m$ .

The second important feature is the connection between the minimizer  $\hat{C}^\infty(P, x)$  of the relaxed problem and minimizing sequences of configurations. To make the connection, the following optimal design problems are introduced. For  $p(x) \geq 0$  and  $p(x)$  in  $C^\infty(\bar{\Omega})$ ,

$$P_j = \inf_{C(P, x) \text{ in } \mathcal{C}(A, \gamma)} F(\phi), \tag{29}$$



subject to the constraint

$$\int_{\Omega} p(x)|\nabla\phi|^2 dx \leq K^2(1 + 1/j), \tag{30}$$

where  $\phi$  is the  $H_0^1(\Omega)$  solution to the state equation

$$-\text{div}(C(\underline{A}, x)\nabla\phi) = f. \tag{31}$$

It is clear that, as  $j$  tends to infinity, the constraints given in the design problems  $P_j$  approach the constraint associated with (7). The problems  $P_j$  share the same feature as (7) in that they are optimal design problems over configurations of  $N$  dielectrics. The connection between the minimizer of the relaxed problem and minimizing sequences of configurations is given in the following theorem.

**Theorem 1.4.** Given a minimizer  $\hat{C}^\infty(\underline{P}, x)$  of  $QP$ , there is a sequence of configurations and associated controls  $\{C^j(\underline{P}, x)\}_{j=1}^\infty$  in  $\mathcal{C}(\underline{A}, \gamma)$  such that, for all  $\underline{P}$  in  $\mathcal{N}(\underline{A})$ , the sequence  $\{C^j(\underline{P}, x)\}_{j=1}^\infty$   $\mathbf{G}$ -converges to  $\hat{C}^\infty(\underline{P}, x)$  and the state variables  $\phi^j$  satisfy the constraints (30). For this case, one has

$$\lim_{j \rightarrow \infty} F(\phi^j) = QP. \tag{32}$$

Moreover, given any  $\epsilon > 0$ , there exists an index  $J > 0$  such that, for all  $j > J$ ,

$$P_j \leq F(\phi^j) \leq P_j + \epsilon \tag{33}$$

and

$$\lim_{j \rightarrow \infty} P_j = QP. \tag{34}$$

Thus, one can use the minimizer for the relaxed problem to recover nearly-optimal constrained-gradient designs of  $N$  dielectrics.

The locally-constrained optimal design problem is given by

$$\inf_{C(\underline{P}, x) \in \mathcal{C}(\underline{A}, \gamma)} F(\phi), \tag{35}$$

where  $\phi$  is a  $H_0^1(\Omega)$  solution of the equation of state (1) and is subject to the gradient constraint given by

$$|\nabla\phi| \leq K, \quad \text{almost everywhere in } \Omega. \tag{36}$$

The local constraint can be given also in the equivalent form (4). Moreover, choosing a countable dense subset  $\{p_k\}_{k=1}^\infty$  of the set of nonnegative functions in  $C^\infty(\bar{\Omega})$ , the local constraint is equivalent to

$$\int_{\Omega} p_k(x) |\nabla \phi|^2 dx \leq K^2 \int_{\Omega} p_k(x) dx, \quad \text{for } 1 \leq k < \infty. \tag{37}$$

The relaxed formulation of (35) is

$$QR = \inf_{C^\infty(\underline{P}, x) \text{ in } \mathcal{C}\mathcal{C}(A, \gamma)} F(\phi), \tag{38}$$

subject to the local constraint

$$\sqrt{(\sigma(x) + I) \nabla \phi \cdot \nabla \phi} \leq K, \quad \text{almost everywhere in } \Omega, \tag{39}$$

where  $\phi$  is the  $H_0^1(\Omega)$  solution to the state equation

$$-\text{div}(C^\infty(A, x) \nabla \phi) = f \tag{40}$$

and  $\sigma$  is defined by (18). The first important feature of the relaxed problem is that the infimum of the design problem is attained by a control in  $\mathcal{C}\mathcal{C}(A, \gamma)$ .

**Theorem 1.5.** There exists a control  $\hat{C}^\infty(\underline{P}, x)$  in  $\mathcal{C}\mathcal{C}(A, \gamma)$  and a state variable  $\hat{\phi}$  in  $H_0^1(\Omega)$  for which

$$-\text{div}(\hat{C}^\infty(A, x) \nabla \hat{\phi}) = f \tag{41}$$

and

$$QR = F(\hat{\phi}), \tag{42}$$

where

$$\sqrt{(\hat{\sigma}(x) + I) \nabla \hat{\phi} \cdot \nabla \hat{\phi}} \leq K, \quad \text{almost everywhere in } \Omega, \tag{43}$$

and

$$\hat{\sigma}(x) E \cdot E = \sum_{i=1}^N (\nabla_{kk}^i \hat{C}^\infty(A, x)) E \cdot E - |E|^2, \tag{44}$$

for every  $E$  in  $R^m$ .

The second important feature is the connection between the minimizer  $\hat{C}^\infty(\underline{P}, x)$  of the relaxed problem and minimizing sequences of configurations. To make the connection, consider a countable dense subset  $\{p_k\}_{k=1}^\infty$  of the nonnegative functions in  $C^\infty(\bar{\Omega})$ . For a finite collection of these functions  $S_j = \{p_k\}_{k=1}^j$ , the following optimal design problems are

introduced:

$$\tilde{P}_j = \inf_{C(\underline{P}, x) \in \mathcal{C}(\underline{A}, \underline{\gamma})} F(\phi), \tag{45}$$

subject to the constraints

$$\int_{\Omega} p_k(x) |\nabla \phi|^2 dx \leq K^2 \int_{\Omega} p_k(x) (1 + 1/j) dx, \quad \text{for } p_k \text{ in } S_j, \tag{46}$$

where  $\phi$  is the  $H_0^1(\Omega)$  solution to the state equation

$$-\text{div}(C(\underline{A}, x)\nabla\phi) = f. \tag{47}$$

It is clear that, as  $j$  tends to infinity, the constraints given in the design problems  $\tilde{P}_j$  approach the constraint given by (37). The problems  $\tilde{P}_j$  share the same feature as (35) in that they are optimal design problems over all configurations of  $N$  dielectrics. The connection between the minimizer of the relaxed problem and minimizing sequences of configurations is given in the following theorem.

**Theorem 1.6.** Given a minimizer  $\hat{C}^\infty(\underline{P}, x)$  of  $QR$ , there is a sequence of configurations and associated controls  $\{C^j(\underline{P}, x)\}_{j=1}^\infty$  in  $\mathcal{C}(\underline{A}, \underline{\gamma})$  such that, for all  $\underline{P}$  in  $\mathcal{N}(\underline{A})$ , the sequence  $\{C^j(\underline{P}, x)\}_{j=1}^\infty$   $\mathbf{G}$ -converges to  $\hat{C}^\infty(\underline{P}, x)$  and the state variables  $\phi^j$  satisfy the constraints (46). For this case, one has

$$\lim_{j \rightarrow \infty} F(\phi^j) = QR. \tag{48}$$

Moreover, given any  $\epsilon > 0$ , there exists an index  $J > 0$  such that, for all  $j > J$ ,

$$\tilde{P}_j \leq F(\phi^j) \leq \tilde{P}_j + \epsilon \tag{49}$$

and

$$\lim_{j \rightarrow \infty} \tilde{P}_j = QR. \tag{50}$$

Thus, one can use the minimizer for the relaxed problem to recover nearly-optimal constrained-gradient designs of  $N$  dielectrics.

The paper is organized as follows. In Section 2, the local formula for the derivative of the effective dielectric tensor viewed as a function of its constituent dielectric constants is derived together with a local homogenization theorem. In Section 3, Theorem 1.2 is derived; the relaxation theorems are proved in Section 4. In Section 5, an overview of the results given

here and recent results for problems where the design criteria are given in terms of the  $L^2$  norm of the gradient are discussed in the conclusion.

**2. Local Formulas for the Derivative of the G-Limit and Local Homogenization**

In this section, we prove Theorem 1.1. We consider a sequence  $\{C^n(\underline{P}, x)\}_{n=1}^\infty$  that G-converges to  $C^\infty(\underline{P}, x)$  for every  $\underline{P}$  in  $\mathcal{N}(\underline{A})$ . Given the sequence  $\{C^n(\underline{A}, x)\}_{n=1}^\infty$ , a vector  $E$  in  $R^m$ , and an open set  $\omega \subset \Omega$ , we introduce the local corrector functions  $\varphi_E^n$ , where  $\varphi_E^n$  is the  $H_0^1(\omega)$  solution of

$$-\text{div}(C^n(\underline{A}, x)(\nabla\varphi_E^n + E)) = -\text{div}(C^\infty(\underline{A}, x)E), \text{ in } \omega. \tag{51}$$

Equation (51) is written in the variational form given by

$$\int_\omega C^n(\underline{A}, x)(\nabla\varphi_E^n + E) \cdot \nabla\psi \, dx - \int_\omega (C^\infty(\underline{A}, x)E) \cdot \nabla\psi \, dx = 0, \tag{52}$$

for all  $\psi$  in  $H_0^1(\omega)$ . In the sequel, a generic open subset of  $\Omega$  is denoted by  $\omega$ . The measure of  $\omega$  is denoted by  $|\omega|$ , and standard a priori estimates using the Cauchy inequality in (52) give

$$\int_\omega |\nabla\varphi_E^n|^2 \, dx \leq (4\Lambda^2/\lambda^2)|\omega||E|^2, \tag{53}$$

$$\int_\omega |\nabla\varphi_E^n + E|^2 \, dx \leq (4\Lambda^2/\lambda^2 + 1)|\omega||E|^2. \tag{54}$$

We now list the convergence properties for the sequence of corrector functions.

**Lemma 2.1.** The sequence  $\{\varphi_E^n\}_{n=1}^\infty$  converges weakly to zero in  $H_0^1(\omega)$ . Thus,

$$\nabla\varphi_E^n \rightharpoonup 0, \quad \text{weakly in } L^2(\omega)^N, \tag{55}$$

and

$$C^n(\underline{A}, x)(\nabla\varphi_E^n + E) \rightharpoonup C^\infty(\underline{A}, x)E, \quad \text{weakly in } L^2(\omega)^N. \tag{56}$$

The proof is standard; see Ref. 10.

For  $\delta\beta$  and  $\underline{M}$  such that  $\underline{A} + \delta\beta\underline{M}$  lies in  $\mathcal{N}(\underline{A})$ , consider the sequence  $\{C^n(\underline{A} + \delta\beta\underline{M}, x)\}_{n=1}^\infty$ . Here,  $\underline{M}$  is taken to be an array of dielectric tensors

that are identically zero except for the  $i$ th component dielectric tensor  $M_i$ . Here,

$$|M_i| = \sqrt{\sum_{k,l} [M_i]_{kl}^2}.$$

The sequence of coefficients  $\{C^n(\underline{A} + \delta\beta \underline{M}, x)\}_{n=1}^\infty$  differs from  $\{C^n(\underline{A}, x)\}_{n=1}^\infty$  by the increment  $\delta\beta M_i \chi_i^n$ . The G-limit for the sequence  $\{C^n(\underline{A} + \delta\beta \underline{M}, x)\}_{n=1}^\infty$  is written as  $C^\infty(\underline{A} + \delta\beta \underline{M}, x)$ . We set

$$\delta C = C^\infty(\underline{A} + \delta\beta \underline{M}, x) - C^\infty(\underline{A}, x),$$

and examine the dependence of  $\delta C$  with respect to  $\delta\beta$ .

The first step is to examine the dependence of  $(1/|\omega|) \int_\omega \delta C E \cdot E \, dx$  with respect to the increment  $\delta\beta$ . The correctors associated with the sequence  $\{C^n(\underline{A} + \delta\beta \underline{M}, x)\}_{n=1}^\infty$  are denoted by  $\hat{\varphi}_E^n + E \cdot x$ , where  $\hat{\varphi}_E^n$  are the  $H_0^1(\omega)$  solutions of

$$\begin{aligned} & \int_\omega C^n(\underline{A} + \delta\beta \underline{M}, x) (\nabla \hat{\varphi}_E^n + E) \cdot \nabla \psi \, dx \\ & - \int_\omega (C^\infty(\underline{A} + \delta\beta \underline{M}, x) E) \cdot \nabla \psi \, dx = 0, \end{aligned} \tag{57}$$

for all  $\psi$  in  $H_0^1(\omega)$ , and

$$C^n(\underline{A} + \delta\beta \underline{M}, x) (\nabla \hat{\varphi}_E^n + E) \rightharpoonup C^\infty(\underline{A} + \delta\beta \underline{M}, x) E, \quad \text{weakly in } L^2(\omega)^N, \tag{58}$$

$$\nabla \hat{\varphi}_E^n \rightharpoonup 0, \quad \text{weakly in } L^2(\omega)^N. \tag{59}$$

For any choice of  $p(x)$  in  $C^\infty(\bar{\omega})$ , it follows from (56) and (58) that

$$\begin{aligned} & (1/|\omega|) \int_\omega p(x) \delta C E \cdot E \, dx \\ & = \lim_{n \rightarrow \infty} (1/|\omega|) \int_\omega p(x) (C^n(\underline{A} + \delta\beta \underline{M}, x) (\nabla \hat{\varphi}_E^n + E) \\ & \quad - C^\infty(\underline{A}, x) (\nabla \varphi_E^n + E)) \cdot E \, dx. \end{aligned} \tag{60}$$

Writing

$$C^n(\underline{A} + \delta\beta \underline{M}, x) = C^n(\underline{A}, x) + \delta\beta M_i \chi_i^n,$$

$$\hat{\varphi}_E^n = \varphi_E^n + \delta\varphi^n,$$

where

$$\delta\varphi^n = \hat{\varphi}_E^n - \varphi_E^n,$$

and substitution into (60) gives

$$\begin{aligned}
 & (1/|\omega|) \int_{\omega} p \delta C E \cdot E \, dx \\
 &= \lim_{n \rightarrow \infty} (1/|\omega|) \int_{\omega} p \delta \beta M_i \chi_i^n (\nabla \varphi_E^n + E) \cdot E \\
 & \quad + p \delta \beta M_i \chi_i^n \nabla \delta \varphi^n \cdot E + p C^n(A, x) \nabla \delta \varphi^n \cdot E \, dx. \tag{61}
 \end{aligned}$$

To proceed further, we subtract (52) from (57) to obtain, for every  $\psi$  in  $H_0^1(\omega)$ , the equation

$$\begin{aligned}
 0 &= \int_{\omega} \delta \beta M_i \chi_i^n (\nabla \varphi_E^n + E) \cdot \nabla \psi \, dx + \int_{\omega} C^n(A, x) \nabla \delta \varphi^n \cdot \nabla \psi \, dx \\
 & \quad + \int_{\omega} \delta \beta M_i \chi_i^n \nabla \delta \varphi^n \cdot \nabla \psi \, dx - \int_{\omega} \delta C E \cdot \nabla \psi \, dx. \tag{62}
 \end{aligned}$$

Choosing  $\psi = (p \varphi_E^n)$  in (62), substitution into (61), and taking limits give

$$\begin{aligned}
 & (1/|\omega|) \int_{\omega} p \delta C E \cdot E \, dx \\
 &= \delta \beta \lim_{n \rightarrow \infty} [(1/|\omega|) \int_{\omega} p M_i \chi_i^n (\nabla \varphi_E^n + E) \cdot (\nabla \varphi_E^n + E) \, dx \\
 & \quad + (1/|\omega|) \int_{\omega} p M_i \chi_i^n (\nabla \varphi_E^n + E) \cdot \nabla \delta \varphi^n \, dx]. \tag{63}
 \end{aligned}$$

Writing

$$\bar{\mathcal{J}} = \limsup_{n \rightarrow \infty} (1/|\omega|) \int_{\omega} p \delta \beta M_i \chi_i^n (\nabla \varphi_E^n + E) \cdot \nabla \delta \varphi^n \, dx, \tag{64a}$$

$$\underline{\mathcal{J}} = \liminf_{n \rightarrow \infty} (1/|\omega|) \int_{\omega} p \delta \beta M_i \chi_i^n (\nabla \varphi_E^n + E) \cdot \nabla \delta \varphi^n \, dx, \tag{64b}$$

standard a priori estimates are used to obtain

$$|\bar{\mathcal{J}}| \leq \|p\|_{\infty} K |E|^2 \sqrt{|\delta \beta|}, \quad |\underline{\mathcal{J}}| \leq \|p\|_{\infty} K |E|^2 \sqrt{|\delta \beta|}, \tag{65}$$

where  $K$  is a constant independent of  $\delta \beta$  and the choice of  $\omega \subset \Omega$  and  $\|p\|_{\infty}$  is the maximum value of  $p$  on  $\bar{\omega}$ . We establish the following lemma.

**Lemma 2.2.** For every  $\omega \subset \Omega$  and function  $p$  in  $C^\infty(\bar{\omega})$ , the limit

$$\lim_{n \rightarrow \infty} [(1/|\omega|) \int_{\omega} p M_i \chi_i^n (\nabla \varphi_E^n + E) \cdot (\nabla \varphi_E^n + E) dx] \tag{66}$$

exists and

$$\begin{aligned} & (1/|\omega|) \int_{\omega} p \delta C E \cdot E dx \\ &= \delta \beta \lim_{n \rightarrow \infty} \left[ (1/|\omega|) \int_{\omega} p M_i \chi_i^n (\nabla \varphi_E^n + E) \cdot (\nabla \varphi_E^n + E) dx \right] + \delta \beta R(\delta \beta, \omega, p), \end{aligned} \tag{67}$$

where

$$|R(\delta \beta, \omega, p)| \leq \|p\|_{\infty} K |E|^2 \sqrt{|\delta \beta|}. \tag{68}$$

**Proof.** We put

$$H_n = (1/|\omega|) \int_{\omega} p M_i \chi_i^n (\nabla \varphi_E^n + E) \cdot (\nabla \varphi_E^n + E) dx, \tag{69a}$$

$$R_n = (1/|\omega|) \int_{\omega} p M_i \chi_i^n (\nabla \varphi_E^n + E) \cdot \nabla \delta \varphi^n dx, \tag{69b}$$

and from (54) we see that

$$\bar{H} = \limsup_{n \rightarrow \infty} H_n \quad \text{and} \quad \underline{H} = \liminf_{n \rightarrow \infty} H_n$$

are finite. Passing to subsequences as necessary, we deduce that

$$(1/|\omega|) \int_{\omega} p \delta C E \cdot E dx = \delta \beta \bar{H} + \delta \beta \bar{R}(\delta \beta, \omega, p), \tag{70}$$

where for each  $\delta \beta$ ,  $\bar{R}(\delta \beta, \omega, p)$  is a cluster point of the sequence  $\{R_n\}_{n=1}^{\infty}$  and from (65) it is evident that

$$\bar{R}(\delta \beta, \omega, p) \leq \|p\|_{\infty} K |E|^2 \sqrt{|\delta \beta|}.$$

Similar considerations give

$$(1/|\omega|) \int_{\omega} p \delta C E \cdot E dx = \delta \beta \underline{H} + \delta \beta \underline{R}(\delta \beta, \omega, p), \tag{71}$$

where  $\underline{R}(\delta \beta, \omega, p)$  is a cluster point of the sequence  $\{R_n\}_{n=1}^{\infty}$  and

$$\underline{R}(\delta \beta, \omega, p) \leq \|p\|_{\infty} K |E|^2 \sqrt{|\delta \beta|}.$$

It is evident from (70) and (71) that  $\underline{H} = \bar{H}$ . From this, we deduce that

$$\underline{R}(\delta\beta, \omega, p) = \bar{R}(\delta\beta, \omega, p) \triangleq R(\delta\beta, \omega, p),$$

and the lemma is established. □

With Lemma 2.2 in hand, we establish Theorem 1.1. We consider subsets  $\omega$  of  $\Omega$  given by cubes  $Q(x, r)$  and the associated local corrector functions are denoted by  $\varphi_E^{n,r}$ . We put

$$H(x, r) = \lim_{n \rightarrow \infty} (1/|Q(x, r)|) \int_{Q(x,r)} M_i \chi_i^n (\nabla \varphi_E^{n,r} + E) \cdot (\nabla \varphi_E^{n,r} + E) dy, \tag{72}$$

and we introduce the functions  $\bar{f}(x)$  and  $\underline{f}(x)$  defined everywhere in  $\Omega$  by

$$\bar{f}(x) \triangleq \limsup_{r \rightarrow 0} H(x, r), \tag{73}$$

$$\underline{f}(x) \triangleq \liminf_{r \rightarrow 0} H(x, r). \tag{74}$$

The estimate (54) shows that these functions are bounded on  $\Omega$ . Given the increment  $\delta\beta$ , we consider the intersection of the Lebesgue points of  $C^\infty(A, x)$  and  $C^\infty(A + \delta\beta M, x)$ . We choose  $p = 1$ , and from Lemma 2.2 we have the following identity:

$$\begin{aligned} \delta C E \cdot E &= \lim_{r \rightarrow 0} (1/|Q(x, r)|) \int_{Q(x,r)} \delta C E \cdot E dy \\ &= \delta\beta \lim_{r \rightarrow 0} (H(x, r) + R(\delta\beta, Q(x, r), 1)), \end{aligned} \tag{75}$$

almost everywhere on  $\Omega$  and for any  $E$  in  $R^m$ . Passing to subsequences if necessary, one has

$$\delta C E \cdot E = \delta\beta(\bar{f}(x) + \bar{R}(\delta\beta, x)), \quad \text{a.e. on } \Omega, \tag{76}$$

where  $\bar{R}(\delta\beta, x)$  is a cluster point of the sequence  $\{R(\delta\beta, Q(x, r), 1)\}_{r>0}$  and

$$|\bar{R}(\delta\beta, x)| \leq K|E|^2 \sqrt{|\delta\beta|}.$$

A similar consideration shows that

$$\delta C E \cdot E = \delta\beta(\underline{f}(x) + \underline{R}(\delta\beta, x)), \quad \text{a.e. on } \Omega, \tag{77}$$

where  $\underline{R}(\delta\beta, x)$  is a cluster point of the sequence  $\{R(\delta\beta, B(x, r), 1)\}_{r>0}$  and

$$|\underline{R}(\delta\beta, x)| \leq K|E|^2 \sqrt{|\delta\beta|}.$$

We subtract (77) from (76) to obtain

$$\bar{f}(x) - \underline{f}(x) = \underline{R}(\delta\beta, x) - \bar{R}(\delta\beta, x), \quad \text{a.e. on } \Omega, \tag{78}$$



to see that

$$|\bar{f}(x) - \underline{f}(x)| < 2K|E|^2\sqrt{|\delta\beta|}, \quad \text{a.e. on } \Omega. \tag{79}$$

Since  $\bar{f}(x)$  and  $\underline{f}(x)$  are independent of the increment  $\delta\beta$ , we deduce that

$$\lim_{r \rightarrow 0} \lim_{n \rightarrow \infty} (1/|Q(x, r)|) \int_{Q(x, r)} M_i \chi_i^n (\nabla \varphi_E^{n,r} + E) \cdot (\nabla \varphi_E^{n,r} + E) dy \tag{80}$$

exists almost everywhere on  $\Omega$ . In this way, it is seen that

$$\underline{R}(\delta\beta, x) = \bar{R}(\delta\beta, x) \triangleq R(\delta\beta, x)$$

and that

$$\begin{aligned} \delta C E \cdot E &= \delta\beta \lim_{r \rightarrow 0} \lim_{n \rightarrow \infty} (1/|Q(x, r)|) \\ &\quad \times \int_{Q(x, r)} M_i \chi_i^n (\nabla \varphi_E^{n,r} + E) \cdot (\nabla \varphi_E^{n,r} + E) dy \\ &\quad + \delta\beta R(\delta\beta, x), \quad \text{a.e. on } \Omega, \end{aligned} \tag{81}$$

where

$$R(\delta\beta, x) < K|E|^2\sqrt{|\delta\beta|}.$$

It is evident from (81) that  $(\partial C^\infty / \partial M_i) E \cdot E$  exists and is given by

$$\begin{aligned} (\partial C^\infty / \partial M_i) E \cdot E &= \lim_{r \rightarrow 0} \lim_{n \rightarrow \infty} (1/|Q(x, r)|) \\ &\quad \int_{Q(x, r)} M_i \chi_i^n (\nabla \varphi_E^{n,r} + E) \cdot (\nabla \varphi_E^{n,r} + E) dy, \end{aligned} \tag{82}$$

and Theorem 1.1 follows.

Collecting results, we substitute (81) into (67) of Lemma 2.3 to obtain the local homogenization theorem.

**Theorem 2.1.** Local Homogenization Theorem. Under the hypotheses of Theorem 1.1, given any  $\omega \subset \Omega$  and  $p$  in  $C^\infty(\bar{\omega})$ , the sequence of local correctors  $\{\varphi_E^n + E \cdot x\}_{n=1}^\infty$  defined by (51) satisfy

$$\begin{aligned} &\lim_{n \rightarrow \infty} (1/|\omega|) \int_{\omega} p M_i \chi_i^n (\nabla \varphi_E^n + E) \cdot (\nabla \varphi_E^n + E) dx \\ &= (1/|\omega|) \int_{\omega} p (\partial C^\infty / \partial M_i) E \cdot E dx. \end{aligned} \tag{83}$$

For future reference, it is noted that the estimate (54) gives the bound

$$\sigma(x)E \cdot E \leq (4\Lambda^2/\lambda^2)|E|^2, \quad (84)$$

almost everywhere in  $\Omega$  for  $E$  in  $R^m$ .

### 3. Homogenization of the Products of Weakly Converging Sequences of Gradients

In this section, Theorem 1.1 and Theorem 2.1 are applied together with a localization argument to establish Theorem 1.2. Recall that the sequences of solutions of (15), denoted by  $\{\phi^n\}_{n=1}^\infty$ , converge weakly to the solution  $\phi^\infty$  of (16) and the associated sequence of coefficient matrices  $\{C^n(\underline{P}, x)\}_{n=1}^\infty$   $G$ -converges to  $C^\infty(\underline{P}, x)$  for every  $\underline{P}$  in  $\mathcal{N}(\underline{A})$ .

The first step in the proof is to approximate  $\phi^\infty$  by piecewise-affine functions. Given  $\epsilon > 0$ , there exists a function  $w^\epsilon$  in  $H_0^1(\Omega)$  which is piecewise affine on  $\Omega$  and

$$\int_{\Omega} |\nabla w^\epsilon - \nabla \phi^\infty|^2 dx < \epsilon^2; \quad (85)$$

see for example Ref. 18. The gradient of  $w^\epsilon$  is constant on the open sets  $\omega_\epsilon^i$  and

$$\bar{\Omega} = \bigcup_{i=1}^{\kappa(\epsilon)} \bar{\omega}_\epsilon^i.$$

In each open set  $\omega_\epsilon^i$ , one has

$$w^\epsilon = E^i \cdot x + c^i,$$

where  $E^i$  is a constant vector in  $R^m$  and  $c^i$  is a constant.

The global corrector  $\{\phi^{n,\epsilon}\}_{n=1}^\infty$  is introduced. The right-hand side

$$f^\epsilon = -\operatorname{div}(C^\infty(\underline{A}, x) \nabla w^\epsilon)$$

is chosen and  $\phi^{n,\epsilon}$  is defined to be the  $H_0^1(\Omega)$  solution of

$$-\operatorname{div}(C^n(\underline{A}, x) \nabla \phi^{n,\epsilon}) = f^\epsilon. \quad (86)$$

One obtains easily the following convergence properties.

**Lemma 3.1.** For  $\epsilon$  fixed,

$$\phi^{n,\epsilon} \rightharpoonup w^\epsilon, \quad \text{as } n \rightarrow \infty, \text{ in } H_0^1(\Omega), \quad (87a)$$

$$C^n(\underline{A}, x) \nabla \phi^{n,\epsilon} \rightharpoonup C^\infty(\underline{A}, x) \nabla w^\epsilon, \quad \text{in } L^2(\Omega)^m. \quad (87b)$$

The error between the weakly converging sequences  $\{\phi^n\}_{n=1}^\infty$  and the sequence of global correctors  $\{\phi^{n,\epsilon}\}_{n=1}^\infty$  can be controlled uniformly with respect to  $n$ . This is made precise in the following lemma.

**Lemma 3.2.** There exists a constant  $C_0$  independent of  $n$  and  $\epsilon$  such that

$$\int_{\Omega} |\nabla\phi^n - \nabla\phi^{n,\epsilon}|^2 dx \leq C_0^2(\Lambda/\lambda)^2 \epsilon^2, \quad \text{for all } n > 0. \tag{88}$$

**Proof.** We start by showing that

$$\|f^\epsilon - f\|_{H^{-1}(\Omega)} \leq \epsilon\Lambda.$$

Given  $\psi$  in  $H_0^1(\Omega)$ , we have

$$\begin{aligned} | \langle_{H^{-1}}(f^\epsilon - f), \psi \rangle_{H_0^1} | &= \left| \int_{\Omega} C^\infty(\underline{A}, x) (\nabla w^\epsilon - \nabla\phi) \cdot \nabla\psi \, dx \right| \\ &\leq \Lambda \int_{\Omega} |\nabla w^\epsilon - \nabla\phi| |\nabla\psi| \, dx \\ &\leq \epsilon\Gamma \|\psi\|_{H_0^1(\Omega)}, \end{aligned} \tag{89}$$

and the claim follows. Next, one has the estimate given by

$$\begin{aligned} \lambda \int_{\Omega} |\nabla\phi^n - \nabla\phi^{n,\epsilon}|^2 dx &\leq \int_{\Omega} C^n(\underline{A}, x) (\nabla\phi^n - \nabla\phi^{n,\epsilon}) \cdot (\nabla\phi^n - \nabla\phi^{n,\epsilon}) \, dx \\ &\leq \|f^\epsilon - f\|_{H^{-1}} C_0 \sqrt{\int_{\Omega} |\nabla\phi^n - \nabla\phi^{n,\epsilon}|^2 dx}, \end{aligned} \tag{90}$$

where  $C_0$  depends only on  $\Omega$  and  $m$  and comes from the Poincaré–Friedrichs inequality. The inequality (88) follows by noting that

$$\|f^\epsilon - f\|_{H^{-1}} \leq \Lambda\epsilon. \tag{91}$$

Collecting results, we write

$$\phi = w^\epsilon + r^\epsilon, \quad \phi^n = \phi^{n,\epsilon} + z^{n,\epsilon}.$$

Here,  $r^\epsilon$  and  $z^{n,\epsilon}$  are in  $H_0^1(\Omega)$  and satisfy

$$\int_{\Omega} |\nabla r^\epsilon|^2 dx < \epsilon^2, \quad \int_{\Omega} |\nabla z^{n,\epsilon}|^2 dx < \epsilon^2, \quad \text{for all } n > 0. \tag{91}$$

Given  $p(x)$  in  $C^\infty(\bar{\Omega})$ , one observes that

$$1 = \sum_{j=1}^N \chi_j^n(x)$$

and writes

$$\begin{aligned} \int_{\Omega} p|\nabla\phi^n|^2 dx &= \int_{\Omega} p\left(\sum_{j=1}^N \chi_j^n(x)\right)|\nabla(\phi^{n,\epsilon} + z^{n,\epsilon})|^2 dx \\ &= \sum_{i=1}^{\kappa(\epsilon)} \sum_{j=1}^N \int_{\omega_\epsilon^i} p\chi_j^n |\nabla\phi^{n,\epsilon}|^2 dx + O(\epsilon). \end{aligned} \tag{92}$$

The lemma that gives the connection between global and local correctors is provided by the following localization result.

**Lemma 3.3.** Localization Lemma. Let  $\phi_{E^i}^{i,n,\epsilon}$  be the  $H_0^1(\omega_\epsilon^i)$  solution of

$$-\operatorname{div}(C^n(\underline{A}, x))(\nabla\phi_{E^i}^{i,n,\epsilon} + E^i) = -\operatorname{div}(C^\infty(\underline{A}, x)E^i). \tag{93}$$

Then,

$$\lim_{n \rightarrow \infty} \int_{\omega_\epsilon^i} |\nabla\phi_{E^i}^{i,n,\epsilon} + E^i - \nabla\phi^{n,\epsilon}|^2 dx = 0. \tag{94}$$

**Proof.** From the properties of the local corrector introduced in Section 2, one has

$$\phi_{E^i}^{i,n,\epsilon} \rightharpoonup 0, \quad \text{weakly in } H_0^1(\omega_\epsilon^i), \tag{95a}$$

$$C^n(\underline{A}, x)(\nabla\phi_{E^i}^{i,n,\epsilon} + E^i) \rightharpoonup C^\infty(\underline{A}, x)E^i, \quad \text{weakly in } L^2(\omega_\epsilon^i)^m. \tag{95b}$$

The properties of the global corrector  $\phi^{n,\epsilon}$  imply that it is a solution of

$$-\operatorname{div}(C^n(\underline{A}, x)\nabla\phi^{n,\epsilon}) = -\operatorname{div}(C^\infty(\underline{A}, x)E^i), \quad \text{on } \omega_\epsilon^i; \tag{96}$$

i.e., for all  $\psi$  in  $H_0^1(\omega_\epsilon^i)$ ,

$$\int_{\omega_\epsilon^i} C^n(\underline{A}, x)\nabla\phi^{n,\epsilon} \cdot \nabla\psi dx = \int_{\omega_\epsilon^i} C^\infty(\underline{A}, x)E^i \cdot \nabla\psi dx, \tag{97}$$

and

$$\phi^{n,\epsilon} \rightharpoonup w^\epsilon = E^i \cdot x + c^i, \quad \text{weakly in } H^1(\omega_\epsilon^i), \tag{98a}$$

$$C^n(\underline{A}, x)\nabla\phi^{n,\epsilon} \rightharpoonup C^\infty(\underline{A}, x)E^i, \quad \text{weakly in } L^2(\omega_\epsilon^i)^m. \tag{98b}$$

The lemma now follows immediately from Remark 1 of Section 10 in Ref. 10. □

The local homogenization Theorem 2.1 and the localization Lemma 3.3 are applied and we work up from small to intermediate scales to find that

$$\begin{aligned}
 & \sum_{i=1}^{\kappa(\epsilon)} \int_{\omega_\epsilon^i} p(\sigma(x) + I) E^i \cdot E^i dx \\
 &= \sum_{i=1}^{\kappa(\epsilon)} \sum_{j=1}^N \int_{\omega_\epsilon^i} p \nabla_{kk}^j C^\infty(\underline{A}, (x)) E^i \cdot E^i dx \\
 &= \sum_{i=1}^{\kappa(\epsilon)} \sum_{i=j}^N \lim_{n \rightarrow \infty} \int_{\omega_\epsilon^i} p \chi_j^n |\nabla \phi_{E^i}^{i,n,\epsilon} + E^i|^2 dx \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^{\kappa(\epsilon)} \sum_{j=1}^N \int_{\omega_\epsilon^i} p \chi_j^n |\nabla \phi^{n,\epsilon}|^2 dx. \tag{99}
 \end{aligned}$$

Proceeding from large scales down to intermediate scales, we see that

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \int_{\Omega} p |\nabla \phi^n|^2 dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^{\kappa(\epsilon)} \int_{\omega_\epsilon^i} p |\nabla \phi^{n,\epsilon}|^2 dx + O(\epsilon) \\
 &= \sum_{i=1}^{\kappa(\epsilon)} \int_{\omega_\epsilon^i} p(\sigma(x) + I) E^i \cdot E^i dx + O(\epsilon) \\
 &= \int_{\Omega} p(\sigma(x) + I) \nabla w^\epsilon \cdot \nabla w^\epsilon dx + O(\epsilon). \tag{100}
 \end{aligned}$$

Since  $\epsilon$  is arbitrary and recalling (84) and (91), we obtain the desired result given by

$$\lim_{n \rightarrow \infty} \int_{\Omega} p |\nabla \phi^n|^2 dx = \int_{\Omega} p(\sigma(x) + I) \nabla \phi^\infty \cdot \nabla \phi^\infty dx, \tag{101}$$

and Theorem 1.2 follows.

#### 4. Relaxation

In this section, the properties of the relaxed problem (38) given by Theorems 1.5 and 1.6 are established. The proofs of Theorems 1.3 and 1.4 follow identical lines. In establishing the properties of the relaxed problem, the following compactness result will be used.

**Theorem 4.1.** Compactness Property. Given any sequence  $\{C^n(\underline{P}, x)\}_{n=1}^\infty$  in  $\mathcal{C}(\underline{A}, \underline{\gamma})$ , there exists a subsequence  $\{C^{n'}(\underline{P}, x)\}_{n'=1}^\infty$  and a

function  $C^\infty(\underline{P}, x)$  in  $\mathcal{CC}(\underline{A}, \gamma)$  such that  $C^n(\underline{P}, x)$  G-converges to  $C^\infty(\underline{P}, x)$  for all  $\underline{P}$  in  $\mathcal{N}(\underline{A})$ .

This property follows immediately from Ref. 13.

A special sequence of controls  $\{C^n(\underline{P}, x)\}_{n=1}^\infty$  in  $\mathcal{C}(\underline{A}, \gamma)$  is constructed. This sequence will be used to establish Theorems 1.5 and 1.6. To construct this sequence, consider a minimizing sequence  $\{C^{\infty,n}(\underline{P}, x)\}_{n=1}^\infty$  for  $QR$ . The associated set of electric potentials for the sequence  $\{C^{\infty,n}(\underline{A}, x)\}_{n=1}^\infty$  is denoted by  $\{\phi^n\}_{n=1}^\infty$  and

$$QR = \lim_{n \rightarrow \infty} F(\phi^n).$$

For each  $n$ , there exists a sequence of controls  $\{C^{n,k}(\underline{P}, x)\}_{k=1}^\infty$  in  $\mathcal{C}(\underline{A}, \gamma)$  such that  $C^{n,k}(\underline{P}, x)$  G-converges to  $C^{\infty,n}(\underline{P}, x)$  for every  $\underline{P}$  in  $\mathcal{N}(\underline{A})$ . The associated potentials  $\phi^{n,k}$  in  $H_0^1(\Omega)$  satisfy the state equation

$$-\text{div}(C^{n,k}(\underline{A}, x)\nabla\phi^{n,k}) = f, \tag{102}$$

and for every nonnegative  $p$  in  $C^\infty(\bar{\Omega})$ , the homogenization Theorem 1.2 gives

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\Omega} p(x)|\nabla\phi^{n,k}|^2 dx &= \int_{\Omega} p(x)(\sigma^n(x) + I)\nabla\phi^n \cdot \nabla\phi^n dx \\ &\leq K^2 \int_{\Omega} p(x) dx, \end{aligned} \tag{103}$$

where the covariance matrix  $\sigma^n(x)$  is

$$\sigma^n(x)E \cdot E = \sum_{i=1}^N (\nabla_{kk}^i C^{\infty,n}(\underline{A}, x))E \cdot E - |E|^2, \tag{104}$$

for all vectors  $E$  in  $R^m$ . A countable dense subset  $\{p_l\}_{l=1}^\infty$  of the set of nonnegative functions in  $C^\infty(\bar{\Omega})$  is introduced. Given  $n$ , put

$$S_n = \{p_l\}_{l=1}^n;$$

then, there exists an index  $k_n$  for which

$$\int_{\Omega} p_l |\nabla\phi^{n,k_n}|^2 dx \leq K^2 \int_{\Omega} p_l (1 + 1/n) dx, \quad \text{for every } p_l \text{ in } S_n, \tag{105}$$

$$|F(\phi^{n,k_n}) - F(\phi^n)| < 1/n, \tag{106}$$

and

$$-\text{div}(C^{n,k_n}(\underline{A}, x)\nabla\phi^{n,k_n}) = f. \tag{107}$$

Passing to a subsequence if necessary and appealing to the compactness property, there exists a function  $\tilde{C}^\infty(\underline{P}, x)$  in  $\mathcal{C}(\underline{A}, \underline{\gamma})$  for which

$$\{C^{n,k_n}(\underline{P}, x)\}_{n=1}^\infty \text{ G-converges to } \tilde{C}^\infty(\underline{P}, x) \tag{108}$$

for all  $\underline{P}$  in  $\mathcal{N}(\underline{A})$ .

**Definition 4.1.** The sequence  $\{(C^{n,k_n}(\underline{P}, x))_{n=1}^\infty$  constructed above satisfying (105)–(108) is called a configuration minimizing sequence.

The potential associated with the control  $\tilde{C}^\infty(\underline{P}, x)$  is denoted by  $\tilde{\phi}$  and

$$-\text{div}(\tilde{C}^\infty(\underline{A}, x)\nabla\tilde{\phi}) = f. \tag{109}$$

We proceed to establish Theorem 1.5. For each  $p_l$  in  $\{p_l\}_{l=1}^\infty$ , it follows that

$$\begin{aligned} \int_{\Omega} p_l(x)(\tilde{\sigma}(x) + I)\nabla\tilde{\phi} \cdot \nabla\tilde{\phi} \, dx &= \lim_{n \rightarrow \infty} \int_{\Omega} p_l |\nabla\phi^{n,k_n}|^2 \, dx \\ &\leq K^2 \int_{\Omega} p_l \, dx, \end{aligned} \tag{110}$$

where

$$\tilde{\sigma}(x)E \cdot E = \sum_{i=1}^N (\nabla_{kk}^i \tilde{C}^\infty(\underline{A}, x))E \cdot E - |E|^2, \tag{111}$$

for every  $E$  in  $R^m$ . Thus, by density, for any nonnegative  $p$  in  $C^\infty(\bar{\Omega})$ , one has

$$\int_{\Omega} p(x)(\tilde{\sigma}(x) + I)\nabla\tilde{\phi} \cdot \nabla\tilde{\phi} \, dx \leq K^2 \int_{\Omega} p(x) \, dx, \tag{112}$$

and  $\tilde{C}^\infty(\underline{P}, x)$  is an admissible control for  $QR$ . It is evident from the continuity of the objective function and (106) that  $F(\tilde{\phi}) = QR$  and Theorem 1.5 follows.

To prove Theorem 1.6, one considers the design problems  $\tilde{P}_j$  given by (45) and starts by showing that  $\tilde{P}_j \leq QR$ . For the configuration minimizing sequence  $\{(C^{n,k_n}(\underline{P}, x))_{n=1}^\infty$  of Definition 4.1, it is evident that, given the index  $j$ , one has that, for all  $n > j$ ,  $C^{n,k_n}(\underline{P}, x)$  is admissible for  $\tilde{P}_j$  and

$$\tilde{P}_j \leq F(\phi^{n,k_n}).$$

Sending  $n$  to infinity shows that

$$\tilde{P}_j \leq QR.$$

Noting that  $\tilde{P}_j$  is monotone increasing with  $j$  and bounded above implies the existence of  $\lim_{j \rightarrow \infty} \tilde{P}_j$ . Next, it is shown that

$$QR = \lim_{j \rightarrow \infty} \tilde{P}_j.$$

Given  $j > 0$ , one can choose a design denoted by  $C^j(\underline{P}, x)$  with associated electric potential  $\phi^j$  in  $H_0^1(\Omega)$  for which

$$\int_{\Omega} p_k(x) |\nabla \phi^j|^2 dx \leq K^2 \int_{\Omega} p_k(x) (1 + 1/j) dx, \quad \text{for } p_k \text{ in } S_j, \quad (113)$$

and

$$-\text{div}(C^j(\underline{A}, x) \nabla \phi^j) = f, \quad (114)$$

where

$$\tilde{P}_j \leq F(\phi^j) \leq \tilde{P}_j + 1/j. \quad (115)$$

From the compactness property and passage to subsequences if necessary, there exists a function  $\tilde{C}^\infty(\underline{P}, x)$  in  $\mathcal{GC}(\underline{A}, \underline{\gamma})$  such that  $C^j(\underline{P}, x)$   $\mathbf{G}$ -converges to  $\tilde{C}^\infty(\underline{P}, x)$  for for all  $\underline{P}$  in  $\mathcal{N}(\underline{A})$ . The associated potential  $\bar{\phi}$  in  $H_0^1(\Omega)$  solves

$$-\text{div}(\tilde{C}^\infty(\underline{A}, x) \nabla \bar{\phi}) = f. \quad (116)$$

Here,

$$\lim_{j \rightarrow \infty} F(\phi^j) = F(\bar{\phi}),$$

and arguing as above, the associated covariance  $\bar{\sigma}$  satisfies

$$\int_{\Omega} p(x) (\bar{\sigma}(x) + I) \nabla \bar{\phi} \cdot \nabla \bar{\phi} dx \leq K^2 \int_{\Omega} p(x) dx, \quad (117)$$

for every  $p$  in  $C^\infty(\bar{\Omega})$ . Thus,  $\tilde{C}^\infty(\underline{P}, x)$  is an admissible design for  $QR$ . Observing that

$$QR \leq F(\bar{\phi}) = \lim_{j \rightarrow \infty} F(\phi^j)$$

and that (115) gives the set of inequalities

$$QR \leq \lim_{j \rightarrow \infty} F(\phi^j) \leq \lim_{j \rightarrow \infty} \tilde{P}_j \leq QR, \quad (118)$$

one concludes that

$$QR = \lim_{j \rightarrow \infty} \tilde{P}_j.$$



Lastly, (49) is established. Consider the configuration minimizing sequence  $\{(C^{n,k_n}(\underline{P}, x))\}_{n=1}^\infty$ . It is evident that the control  $C^{j,k_j}(\underline{P}, x)$  is admissible for  $\tilde{P}_j$ ; thus,

$$P_j \leq F(\phi^{j,k_j}). \tag{119}$$

The upper inequality in (49) follows recalling that

$$\lim_{j \rightarrow \infty} F(\phi^{j,k_j}) = QR,$$

where  $\phi^{j,k_j}$  is the state variable associated with  $C^{j,k_j}(\underline{A}, x)$  and on noting that

$$\lim_{j \rightarrow \infty} \tilde{P}_j = QR.$$

### 5. Conclusions

The results given here are intended to provide the underpinnings for numerical methods for the design of multiphase composites for optimal performance subject to constraints on the gradient of the state variable. Future work will apply the theory developed here to the numerical design of functionally-graded materials.

It should be pointed out that, for the case of two-phase mixtures, with one component having a nonzero dielectric constant and the other being zero, the comprehensive theory developed in Ref. 19 provides a relaxed formulation for the design problem when the constraint is given in terms of the  $L^2$ -norm of the gradient. Here, the fact that one of the dielectrics is zero delivers an optimal design problem of self-adjoint type. However, when both dielectric constants are nonzero, the problem becomes nonself-adjoint and a design problem of the kind treated here is recovered.

Recently, problems involving design criteria given directly in terms of the  $L^2$ -norm of the gradient have been addressed (Refs. 20–25). Here, the objective is to force the gradient to be as close as possible to a target field in the  $L^2$  norm. This problem is one of minimizing the mean-square deviation of the gradient of the state. Gradient-based methods for the numerical identification of minimizing sequences of configurations are given in Ref. 20, while the full relaxation for the problem has been found recently in Ref. 21. In Ref. 22, a philosophy similar to Ref. 19 is adopted, and the new concept of constrained quasiconvex envelope is introduced. The constrained quasiconvex envelope is computed explicitly in Ref. 23 for this problem. It is shown in Ref. 23 that the only minimizing sequences for the minimization of the mean-square deviation of the gradient of the state are the well-known

rank-one layered microstructures. The earlier work given in Ref. 24 indicates the existence of a dense class of targets for which the relaxation is accomplished through computation of the strong  $L^2$ -closure of the set of controls. The explicit representation of this set is given in Ref. 24. For the case of  $N$  anisotropic conductors, the strong  $L^2$  closure is given in Ref. 25.

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