Optimal bounds on effective elastic tensors for orthotropic composites†

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We consider the totality of orthotropic composites made from two isotropic linearly elastic components in fixed proportion. The elastic properties of orthotropic composites are characterized by nine independent moduli. We provide bounds for six of these, namely the three Youngs moduli and three in-plane shear moduli. The bounds are optimal and correlate the six moduli.

1. Introduction

Recent investigations have focused on the characterization of composites with prescribed symmetry. Work in this area provide bounds on the moduli of elastic tensors associated with composites of a given symmetry class (see Avellaneda 1987; Francfort & Murat 1986; Hashin & Shtrikman 1962; Hashin & Rosen 1964; Hill 1964; James *et al.* 1990; Lipton 1992; Lipton & Northrup 1993; Milton 1985; Norris 1985). In this paper we derive new optimal bounds on six of the nine effective elastic moduli for two-phase orthotropic elastic composites.

We consider orthotropic composites made from two well ordered isotropic elastic components in specified volume fractions. The component elasticities are specified by the tensors C_i, $i = 1, 2$, given by

$$C_i = 2\mu_i \boldsymbol{I} + (\kappa_i - \tfrac{2}{3}\mu_i) I \otimes I \tag{1.1}$$

with $\boldsymbol{I}$ being the identity on 3×3 matrices and I the 3×3 identity matrix. We adopt the convention $\mu_1 \leqslant \mu_2$, $\kappa_1 \leqslant \kappa_2$. The volume fraction of each material in the composite is given by θ_1 for material-1 and θ_2 for material-2 such that $1 = \theta_1 + \theta_2$. The effective compliance tensor $\mathcal{C}^{e-1}$ of an orthotropic composite has the following matrix representation relative to the standard engineering basis of 3×3 strains (cf. Thurston 1984) given by

$$\begin{bmatrix} 1/E_1^e & -\nu_{12}/E_2^e & -\nu_{13}/E_3^e & 0 & 0 & 0 \\ -\nu_{21}/E_1^e & 1/E_2^e & -\nu_{23}/E_3^e & 0 & 0 & 0 \\ -\nu_{31}/E_1^e & -\nu_{32}/E_2^e & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{23}^{e-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{13}^{e-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{12}^{e-1} \end{bmatrix}. \tag{1.2}$$

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Here E_1^e, E_2^e, E_3^e are the effective Youngs moduli, G_{12}^e, G_{13}^e, G_{23}^e are the effective shear moduli, and 'ν_{ij}' are the effective Poisson ratios, with $(\nu_{12}/E_2^e) = (\nu_{21}/E_1^e)$, $(\nu_{13}/E_3^e) = (\nu_{31}/E_1^e)$, and $(\nu_{23}/E_3^e) = (\nu_{32}/E_2^e)$.

In the work of Lipton & Northrup (1993), arithmetic and harmonic mean bounds on the effective shear moduli G_{12}^e, G_{13}^e, G_{23}^e were established, as well as optimal upper and lower bounds on sums of energies associated with shear strains. In this paper we shall expand on these results and obtain optimal bounds correlating the six 'diagonal' moduli E_1^e, E_2^e, E_3^e, G_{12}^e, G_{13}^e and G_{23}^e. In addition we display a method for calculating optimal upper and lower bounds on sums of energies associated with an arbitrary ensemble of stress fields. In physical terms our results provide optimal microstructures that represent the stiffest or most compliant response.

Our method makes exclusive use of the Hashin–Shtrikman variational principles (see Hashin & Shtrikman 1962). It is known in many contexts that the associated upper and lower bounds are saturated by finite rank stiff and compliant laminar composites (see Avellaneda 1987; Gibianskii & Cherkaev 1984; Kohn & Lipton 1988; Milton & Kohn 1988). The extremal properties of laminates are elucidated in the comparison principle given by Avellaneda (1987). In this context the principle states that: given the effective compliance C^{e-1} of an orthotropic composite made from well-ordered components C_1, C_2 in proportions θ_1, θ_2 respectively, then there exists a finite rank orthotropic soft-laminate with compliance $\underline{C}^{-1}$ and a finite rank orthotropic stiff laminate with compliance $\overline{C}^{-1}$ made from the same components and volume fractions such that for any set of 3×3 constant stress tensors σ_i, $i = 1, \ldots, n$.

$$\sum_{i=1}^{n} \overline{C}^{-1} \sigma_i : \sigma_i \leqslant \sum_{i=1}^{n} C^{e^{-1}} \sigma_i : \sigma_i \leqslant \sum_{i=1}^{n} \underline{C}^{-1} \sigma_i : \sigma_i. \tag{1.3}$$

This principle suggests that explicit bounds on the effective elastic moduli for orthotropic composites can be found if one has a closed form description of the set of effective orthotropic compliance tensors $\underline{C}^{-1}$ and $\overline{C}^{-1}$. Indeed, such a description is presented in § 3. This description is applied in § 4 to provide optimal bounds on the diagonal moduli for orthotropic composites.

2. Orthotropic finite rank laminates

A finite rank laminate is defined iteratively. To illustrate, we show how to construct a rank 2 laminate. One starts with a core of material 2 and layers it with a coating of material 1 in layers of thickness ε^2 perpendicular to a specified direction n_1. One then takes this finely layered material and again layers it with a coating of material 1 in layers of thickness ε perpendicular to a second direction n_2. The $\varepsilon \to 0$ limit of this microgeometry is called a rank 2 laminate. Conversely one could start with a core of material 1 and layer it with a coating of material 2 and so on. Laminates of higher rank are constructed in the same way. Explicit formulas have been developed for tensors describing the effective properties of finite rank laminates (see Francfort & Murat 1986; Lurie & Cherkaev 1984; Tartar 1985). For fixed volume fractions θ_1 and θ_2 of materials 1 and 2 the effective elasticity tensor of a rank j strong laminate $\overline{C}$ with material 1 as core and material 2 as layers with layer direction given by the unit vectors $n^1, n^2, \ldots, n^j$ is given

$$\theta_1 \left(\mathcal{C}_2 - \overline{\mathcal{C}}\right)^{-1} = (\mathcal{C}_2 - \mathcal{C}_1)^{-1} - \theta_2 \overset{*}{\mathcal{T}}{}^2 . \tag{2.1}$$

effective elasticity of a rank j compliant laminate $\underline{\mathcal{C}}$ with material 2 as ıd material 1 as layers with layering directions n^1, $n^2, \ldots, n^j$ is given by

$$\theta_2 (\underline{\mathcal{C}} - \mathcal{C}_1)^{-1} = (\mathcal{C}_2 - \mathcal{C}_1)^{-1} + \theta_1 \overset{*}{\mathcal{T}}{}^1 . \tag{2.2}$$

$$\overset{*}{\mathcal{T}}{}^s = \sum_{i=1}^{j} \rho_j \hat{\Gamma}^s(n^i), \quad s = 1, 2 \tag{2.3}$$

$$0 \leqslant \rho_i \leqslant 1, \quad \sum_{i=1}^{j} \rho_i = 1 \tag{2.4}$$

e tensor $\hat{\Gamma}^s(v)$ is given by

$$v)M = \frac{3}{3\kappa_s + 4\mu_s}(M : vv)vv + \frac{1}{\mu_s}[(Mv)v + v(Mv) - (M : vv)vv] \tag{2.5}$$

symmetric 3×3 matrices M and $s = 1, 2$. Here all quantities vv, $(Mv)v$, are the usual dyadic products between vectors. The quantities $\theta_1 \rho_i$ and, ›pearing in (2.1)–(2.3) are the relative proportions of layer materials intro- in the ith lamination. Formulas (2.1) and (2.2) were developed by Francfort at (1986).

llows from equations (2.1) and (2.2) that the effective tensors of finite rank tes are orthotropic if and only if the tensors

$$\overset{*}{\mathcal{T}}{}^s, \quad s = 1, 2$$

hotropic and have matrices of the form given by

$$\begin{matrix} t^s_{11} & t^s_{12} & t^s_{13} & 0 & 0 & 0 \\ t^s_{12} & t^s_{22} & t^s_{23} & 0 & 0 & 0 \\ t^s_{13} & t^s_{23} & t^s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & g^s_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & g^s_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & g^s_1 \end{matrix} \tag{2.6}$$

matrix representation of the orthotropic group (denoted by O) has four ts Q^Γ, $\Gamma = 1, 2, 3, 4$. The identity is denoted by Q^1 and the remaining three n matrices are associated with a rotation of π radians around the i axis, ion of π radians around the j axis and a rotation of π radians around the respectively. Group averaging formula (2.3) over the orthotropic group we explicit formulas for the entries in the matrix (2.6) for $\overset{*}{\mathcal{T}}{}^s$. Indeed these ıs are given by

$$t^s_{mn} = \sum_{i=1}^{j} \rho_i t^s_{mn}(n^i) \text{ for } n \leqslant m = 1, 2, 3, \tag{2.7}$$

$$g_k^s = \sum_{i=1}^{j} \rho_i g_k^s(n^i), \quad k = 1,2,3, \tag{2.8}$$

where the weights $\rho_i \sum \rho_i = 1$, and unit vectors n^i are associated with layer directions and relative layer thicknesses.

Here $t_{mn}^s(v)$ and $g_k^s(v)$ are orthotropic polynomials of degree four on the unit sphere given by

$$\left.\begin{aligned}
t_{11}^s(v) &= (\gamma_s - 2\beta_s)v_1^4 + 2\beta_s v_1^2,\\
t_{22}^s(v) &= (\gamma_s - 2\beta_s)v_2^4 + 2\beta_s v_2^2,\\
t_{33}^s(v) &= (\gamma_s - 2\beta_s)v_3^4 + 2\beta_s v_3^2,\\
t_{12}^s(v) &= (\gamma_s - 2\beta_s)v_1^2 v_2^2,\\
t_{13}^s(v) &= (\gamma_s - 2\beta_s)v_1^2 v_3^2,\\
t_{23}^s(v) &= (\gamma_s - 2\beta_s)v_2^2 v_3^2,\\
g_1^s(v) &= 2(\gamma_s - 2\beta_s)v_1^2 v_2^2 + \beta_s(v_1^2 + v_2^2),\\
g_2^s(v) &= 2(\gamma_s - 2\beta_s)v_1^2 v_3^2 + \beta_s(v_1^2 + v_3^2),\\
g_3^s(v) &= 2(\gamma_s - 2\beta_s)v_2^2 v_3^2 + \beta_s(v_2^2 + v_3^2),
\end{aligned}\right\} \tag{2.9}$$

where β_s and γ_s are the characteristic combinations of constants given by

$$\beta_s = \frac{1}{2\mu_s}, \quad \gamma_s = \frac{3}{3k_s + 4\mu_s}. \tag{2.10}$$

3. The set of effective tensors for orthotropic laminar composites

To obtain a closed form description of the set of effective tensors of orthotropic laminates we determine the set of all parameters t_{mn}^s and g_i^s given by (2.7) and (2.8) as one varies over all layer directions n^i and parameters ρ_i.

Noting that $\sum_{i=1}^{j} \rho_i = 1$ it follows from (2.7)–(2.9) that the parameters t_{mn}^s and g_k^s can be written in terms of moments of a probability measure supported on the unit sphere. To facilitate the subsequent analysis we use the identity $v_1^2 + v_2^2 + v_3^2 = 1$ to rewrite the functions $t_{11}^s(v)$, $t_{22}^s(v)$, $t_{33}^s(v)$, $g_1^s(v)$, $g_2^s(v)$, $g_3^s(v)$ as homogeneous polynomials of degree 4 on the unit sphere. We obtain the formulas

$$\left.\begin{aligned}
t_{11}^s(v) &= (\gamma_s - 2\beta_s)v_1^4 + 2\beta_s(v_1^4 + v_1^2 v_2^2 + v_1^2 v_3^2),\\
t_{22}^s(v) &= (\gamma_s - 2\beta_s)v_2^4 + 2\beta_s(v_1^2 v_2^2 + v_2^4 + v_2^2 v_3^2),\\
t_{33}^s(v) &= (\gamma_s - 2\beta_s)v_3^4 + 2\beta_s(v_1^2 v_3^2 + v_2^2 v_3^2 + v_3^4),\\
g_1^s(v) &= 2(\gamma_s - 2\beta_s)v_1^2 v_2^2 + \beta_s(v_1^4 + 2v_1^2 v_2^2 + v_1^2 v_3^2 + v_2^4 + v_2^2 v_3^2),\\
g_2^s(v) &= 2(\gamma_s - 2\beta_s)v_1^2 v_3^2 + \beta_s(v_1^4 + v_1^2 v_2^2 + 2v_1^2 v_3^2 + v_2^2 v_3^2 + v_3^4),\\
g_3^s(v) &= 2(\gamma_s - 2\beta_s)v_2^2 v_3^2 + \beta_s(v_1^2 v_2^2 + v_2^4 + 2v_2^2 v_3^3 + v_1^2 v_3^2 + v_3^4).
\end{aligned}\right\} \tag{3.1}$$

For any probability measure μ defined on the unit sphere we introduce the moments

$$\left.\begin{aligned}
m_1 &= \int v_1^4\,\mathrm{d}\mu, \quad m_2 = \int v_2^4\,\mathrm{d}\mu, \quad m_3 = \int v_3^4\,\mathrm{d}\mu,\\
m_4 &= \int v_1^2 v_2^2\,\mathrm{d}\mu, \quad m_5 = \int v_1^2 v_3^2\,\mathrm{d}\mu, \quad m_6 = \int v_2^2 v_3^2\,\mathrm{d}\mu,
\end{aligned}\right\} \tag{3.2}$$

integration is over the unit sphere. Noting that the sums in (2.7) and (2.8) ɔond to discrete measures on the unit sphere and since the set of discrete res is dense we see that the parameters t^s_{mn} and g^s_i may be given by

$$\left.\begin{aligned}
t^s_{11} &= (\gamma_s - 2\beta_s)m_1 + 2\beta_s(m_1 + m_4 + m_5),\\
t^s_{22} &= (\gamma_s - 2\beta_s)m_2 + 2\beta_s(m_4 + m_2 + m_6),\\
t^3_{33} &= (\gamma_s - 2\beta_s)m_3 + 2\beta_s(m_5 + m_6 + m_3),\\
t^s_{12} &= (\gamma_s - 2\beta_s)m_4,\\
t^s_{13} &= (\gamma_s - 2\beta_s)m_5,\\
t^s_{23} &= (\gamma_s - 2\beta_s)m_6,\\
g^s_1 &= 2(\gamma_s - 2\beta_s)m_4 + \beta_s(m_1 + 2m_4 + m_5 + m_2 + m_6),\\
g^s_2 &= 2(\gamma_s - 2\beta_s)m_5 + \beta_s(m_1 + m_4 + 2m_5 + m_6 + m_3),\\
g^s_3 &= 2(\gamma_s - 2\beta_s)m_6 + \beta_s(m_4 + m_2 + 2m_6 + m_5 + m_3).
\end{aligned}\right\} \tag{3.3}$$

ɔbserve that a closed from description for the set of effective elastic tensors ; from a closed form characterization of the moments m_i, $i = 1$ to 6. This ɔtion is given in the following theorem.

ːorem 3.1. *The set of moments given by (3.2), associated with all proba- ɱeasures on the unit sphere are precisely all points*

$$\underline{m} \equiv (m_1, m_2, m_3, m_4, m_5, m_6)$$

closed convex set $\mathcal{R} \subset \Re^6$ described by

$$m_i = b_i/r \quad i = 1, 2, \ldots, 6, \tag{3.4}$$

$$r = b_1 + b_2 + b_3 + 2(b_4 + b_5 + b_6)$$

, $i = 1, \ldots, 6$ *lie in the set given by*

$$b_1 \geqslant 0, \quad b_4 \geqslant 0, \quad b_5 \geqslant 0, \quad b_6 \geqslant 0, \tag{3.5}$$

$$\begin{vmatrix} b_1 & b_4 \\ b_4 & b_2 \end{vmatrix} \geqslant 0, \tag{3.6}$$

$$\begin{vmatrix} b_1 & b_4 & b_5 \\ b_4 & b_2 & b_6 \\ b_5 & b_6 & b_3 \end{vmatrix} \geqslant 0. \tag{3.7}$$

ɔf. We denote by S the surface described by the system

$$\{v_1^4, v_2^4, v_3^4, v_1^2v_2^2, v_1^2v_3^2, v_2^2v_3^2\}$$

ints $\underline{v} = (v_1, v_2, v_3)$ on the unit sphere. This surface lies on the plane $+ v_3^4 + 2(v_1^2v_2^2 + v_1^2v_3^2 + v_2^2v_3^2) = 1$, and the closed conic hull $K(S)$ of the : is given by all points $(b_1, b_2, \ldots, b_6)$ defined by

$$\left.\begin{aligned}
b_1 &= \int v_1^4\,\mathrm{d}\sigma, \quad b_2 = \int v_2^4\,\mathrm{d}\sigma, \quad b_3 = \int v_3^4\,\mathrm{d}\sigma,\\
b_4 &= \int v_1^2v_2^2\,\mathrm{d}\sigma, \quad b_5 = \int v_1^2v_3^2\,\mathrm{d}\sigma, \quad b_6 = \int v_2^2v_3^2\,\mathrm{d}\sigma,
\end{aligned}\right\} \tag{3.8}$$

σ is any positive measure on the unit sphere.

In what follows we show that $K(S)$ is precisely the set of points that satisfy (3.5)–(3.7). To see this we note that the dual cone $K^T(S)$ is identified with the set of all non-negative, homogeneous, polynomials of degree 4, invariant under the orthotropic group, i.e. of the type

$$P(\underline{v}) = \alpha v_1^4 + \beta v_2^4 + \delta v_3^4 + \gamma v_1^2 v_2^2 + \rho v_1^2 v_3^2 + \xi v_2^2 v_3^2, \tag{3.9}$$

Moreover, it is immediate that a point $(b_1, \ldots, b_6)$ lies in $K(S)$ if and only if for every non-negative polynomial of the type (3.9); the associated form

$$\alpha b_1 + \beta b_2 + \delta b_3 + \gamma b_4 + \rho b_5 + \xi b_6 \tag{3.10}$$

is non-negative.

Thus as in the theory of the trigonometric and power moment problems (cf. Kreĭn & Nudel'man 1977) a closed form description of $K(S)$ follows from an explicit representation of the dual cone, i.e. the explicit representation of all homogeneous non-negative orthotropic polynomials of degree four on the unit sphere. Fortunately such a representation follows from a theorem of Hilbert (1888) which states that: every positive definite homogeneous polynomial 'F' of degree 4 on $\Re^3$ admits the representation

$$F(x) = \sum_{i=1}^{3} (M^i x \cdot x)^2, \tag{3.11}$$

where M^i, $i = 1, 2, 3$ are symmetric 3×3 matrices and x is in $\Re^3$. From homogeneity it is evident that every positive definite polynomial on the unit sphere also has the representation (3.11). Upon group averaging (3.11) over the orthotropic group (i.e. computing $\frac{1}{4}\sum_{\gamma=1}^{4} F(Q^\gamma v), v$ on the unit sphere) it follows that all non-negative orthotropic homogeneous polynomials of degree 4 on the unit sphere admit the representation

$$P(\underline{v}) = \sum_{i=1}^{3} \{(m_{11}^i v_1^2 + m_{22}^i v_2^2 + m_{33}^i v_3^2)^2 + 4((m_{12}^i)^2 v_1^2 v_2^2 + (m_{13}^i)^2 v_1^2 v_3^2 + (m_{23}^i)^2 v_2^2 v_3^2)\}. \tag{3.12}$$

Here m_{rs}^i are the elements of the matrices M^i. It now follows from (3.12), that points $(b_1, \ldots, b_6)$ lie in $K(S)$ if and only if the form

$$\sum_{i=1}^{3} B\underline{m}^i \cdot \underline{m}^i \tag{3.13}$$

is non-negative for all vectors $\underline{m}^i$. Here $\underline{m}^i \equiv (m_{11}^i, m_{22}^i, m_{33}^i, m_{23}^i, m_{13}^i, m_{12}^i)$, $i = 1, 2, 3$ and the matrix B is given by

$$\begin{bmatrix} b_1 & b_4 & b_5 & 0 & 0 & 0 \\ b_4 & b_2 & b_6 & 0 & 0 & 0 \\ b_5 & b_6 & b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4b_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4b_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4b_4 \end{bmatrix}. \tag{3.14}$$

We observe that inequalities (3.5)–(3.7) follow immediately from the positive semidefiniteness of the matrix B.

obtain the representation for the moments m_i, $i = 1, \ldots, 6$, we apply the ity $(v_1^2 + v_2^2 + v_3^2)^2 = 1$ to obtain the identity

$$r \equiv b_1 + b_2 + b_3 + 2(b_4 + b_5 + b_6) = \int \mathrm{d}\sigma \tag{3.15}$$

3.4) follows.

s evident from Theorem 3.1 that the set of moments $\mathcal{R}$ consists of all points on the intersection of the plane $b_1 + b_2 + b_3 + 2(b_4 + b_5 + b_6) = 1$ and the x set described by (3.5)–(3.7).

see that all moments are attained by finite rank orthotropic laminates we rom Caratheodory's theorem that the convex hull of the surface S can be ssed as

$$\left.\begin{aligned} m_1 = \sum_{j=1}^{k} \rho_j (n_1^j)^4, \quad m_2 = \sum_{j=1}^{k} \rho_j (n_2^j)^4, \quad m_3 = \sum_{j=1}^{k} \rho_j (n_3^j)^4, \\ = \sum_{j=1}^{k} \rho_j (n_1^j)^2 (n_2^j)^2, \quad m_5 = \sum_{j=1}^{k} \rho_j (n_1^j)^2 (n_3^j)^2, \quad m_6 = \sum_{j=1}^{k} \rho_j (n_2^j)^2 (n_3^j)^2, \end{aligned}\right\} \tag{3.16}$$

$k \leqslant 6$ and $\sum_{j=1}^{k} \rho_j = 1$. Attainability follows by observing that substitution ation (3.16) into equation (3.3) corresponds exactly to equations (2.7)–(2.9) n the description of the effective tensor of finite rank laminates.

4. Optimal bounds for diagonal moduli

start by providing explicit closed form formulas for the effective Young's li and in-plane shear moduli of finite rank laminates. Algebraic manipulation ations (2.1), (2.2) and (2.6) shows that the effective Young's moduli for stiff rank orthotropic composites are given by

$$\overline{E}_1 = [c_2 + \theta_1 \{c_2^2 \tilde{A}_{11}^2 + d_2^2(\tilde{A}_{22}^2 + \tilde{A}_{33}^2 + 2\tilde{A}_{23}^2) + 2c_2 d_2(\tilde{A}_{12}^2 + \tilde{A}_{13}^2)\}]^{-1}, \tag{4.1}$$

$$\overline{E}_2 = [c_2 + \theta_1 \{c_2^2 \tilde{A}_{22}^2 + d_2^2(\tilde{A}_{11}^2 + \tilde{A}_{33}^2 + 2\tilde{A}_{13}^2) + 2c_2 d_2(\tilde{A}_{12}^2 + \tilde{A}_{23}^2)\}]^{-1}, \tag{4.2}$$

$$\overline{E}_3 = [c_2 + \theta_1 \{c_2^2 \tilde{A}_{33}^2 + d_2^2(\tilde{A}_{11}^2 + \tilde{A}_{22}^2 + 2\tilde{A}_{12}^2) + 2c_2 d_2(\tilde{A}_{13}^2 + \tilde{A}_{23}^2)\}]^{-1}, \tag{4.3}$$

the constants c_2 and d_2 are given by

$$c_2 = \tfrac{1}{3}(\tfrac{1}{3}\kappa_2^{-1} + \mu_2^{-1}), \quad d_2 = \tfrac{1}{3}(\tfrac{1}{3}\kappa_2^{-1} - \tfrac{1}{2}\mu_2^{-1})$$

$\tilde{A}_{11}^2$, $\tilde{A}_{22}^2$, $\tilde{A}_{33}^2$, $\tilde{A}_{12}^2$, $\tilde{A}_{13}^2$, $\tilde{A}_{23}^2$ are rational functions of the moments described eorem 3.1. (Explicit formulas for $\tilde{A}_{ij}^2$ are given in the Appendix.)

effective in-plane shear moduli for orthotropic finite rank stiff laminates ven by

$$\overline{G}_{23} = 2\mu_2 - \theta_1 2\Delta\mu[1 - \theta_2 2\Delta\mu g_3^2]^{-1}, \tag{4.4}$$

$$\overline{G}_{13} = 2\mu_2 - \theta_1 2\Delta\mu[1 - \theta_2 2\Delta\mu g_2^2]^{-1}, \tag{4.5}$$

$$\overline{G}_{12} = 2\mu_2 - \theta_1 2\Delta\mu[1 - \theta_2 2\Delta\mu g_1^2]^{-1}, \tag{4.6}$$

$\Delta\mu = (\mu_2 - \mu_1)$ and g_1^2, g_2^2 and g_3^2 are functions of the moments given by

(3.3). Effective Youngs moduli for soft orthotropic laminates are given by

$$\underline{E}_1 = [c_1 - \theta_2\{c_1^2\tilde{A}_{11}^1 + d_1^2(\tilde{A}_{22}^1 + \tilde{A}_{33}^1 + 2\tilde{A}_{23}^1) + 2c_1d_1(\tilde{A}_{12}^1 + \tilde{A}_{13}^1)\}]^{-1}, \quad (4.7)$$
$$\underline{E}_2 = [c_1 - \theta_2\{c_1^2\tilde{A}_{22}^1 + d_1^2(\tilde{A}_{11}^1 + \tilde{A}_{33}^1 + 2\tilde{A}_{13}^1) + 2c_1d_1(\tilde{A}_{12}^1 + \tilde{A}_{23}^1)]^{-1}, \quad (4.8)$$
$$\underline{E}_3 = [c_1 - \theta_2\{c_1^2\tilde{A}_{33}^1 + d_1^2(\tilde{A}_{11}^1 + \tilde{A}_{22}^1 + 2\tilde{A}_{12}^1) + 2c_1d_1(\tilde{A}_{13}^1 + \tilde{A}_{23}^1)]^{-1}, \quad (4.9)$$

where $c_1 = \frac{1}{3}(\frac{1}{3}\kappa_1^{-1} + \mu_1^{-1})$; $d_1 = \frac{1}{3}(\frac{1}{3}\kappa_1^{-1} - \frac{1}{2}\mu_1^{-1})$ and $\tilde{A}_{ij}^1$ are rational functions given in the Appendix.

The effective in-plane shear moduli for soft laminates are

$$\underline{G}_{23} = 2\mu_1 + \theta_2 2\Delta\mu[1 + \theta_1 2\Delta\mu g_3^1]^{-1}, \quad (4.10)$$
$$\underline{G}_{13} = 2\mu_1 + \theta_2 2\Delta\mu[1 + \theta_1 2\Delta\mu g_2^1]^{-1}, \quad (4.11)$$
$$\underline{G}_{12} = 2\mu_1 + \theta_2 2\Delta\mu[1 + \theta_1 2\Delta\mu g_1^1]^{-1}. \quad (4.12)$$

Here g_1^1, g_2^1, g_3^1 are functions of the moments given by (3.3).

It is seen from (3.4) that the moments m_i, $i = 1, \ldots, 6$ lie in the plane

$$m_1 + m_2 + m_3 + 2(m_4 + m_5 + m_6) = 1 \quad (4.13)$$

and it follows that the closed convex set R delivered by Theorem 3.1 lies in a five-dimensional affine space. Thus the sets of moduli given by

$$\overline{S} = (\overline{E}_1, \overline{E}_2, \overline{E}_3, \overline{G}_{23}, \overline{G}_{13}, \overline{G}_{12}), \quad (4.14)$$
$$\underline{S} = (\underline{E}_1, \underline{E}_2, \underline{E}_3, \underline{G}_{23}, \underline{G}_{13}, \underline{G}_{12}) \quad (4.15)$$

sweep out surfaces in the six-dimensional space of effective elastic moduli

$$(E_1^e, E_2^e, E_3^e, G_{23}^e, G_{13}^e, G_{12}^e)$$

as the moments m_i, $i = 1, 2, \ldots, 6$ vary over the set R. From the comparison principle we see that these surfaces possess extremal properties in the space of effective moduli. Indeed it follows that

Theorem 4.1. *For any orthotropic composite made from materials C_1, C_2 in the proportions θ_1, θ_2, respectively, then there exists finite rank stiff, and soft orthotropic composites made of the same components and proportions with effective moduli in the sets $\overline{S}$ and $\underline{S}$ such that*

$$\underline{E}_i \leqslant E_i^e \leqslant \overline{E}_i, \quad i = 1, 2, 3, \quad (4.16)$$

$$\underline{G}_{23} \leqslant G_{23}^e \leqslant \overline{G}_{23}, \quad \underline{G}_{13} \leqslant G_{13}^e \leqslant \overline{G}_{13}, \quad \underline{G}_{12} \leqslant G_{12}^e \leqslant \overline{G}_{12}. \quad (4.17)$$

It is evident that the extremal surfaces $\overline{S}$ and $\underline{S}$ given in parametric form by (4.1)–(4.6) and (4.7)–(4.12), deliver bounds that correlate extremal values of the six diagonal moduli.

Bounds on each modulus independent of the other moduli are obtained by finding extremal values for the functions (4.1)–(4.12) as $\underline{m}$ ranges over R. Indeed we let

$$E_i^- = \min_{\underline{m}\in R} \underline{E}_i, \quad E_i^+ = \max_{\underline{m}\in R} \overline{E}_i, \quad i = 1, 2, 3, \quad (4.18)$$

$$G_{23}^- = \min_{\underline{m}\in R} \underline{G}_{23}, \quad G_{13}^- = \min_{\underline{m}\in R} \underline{G}_{13}, \quad G_{12}^- = \min_{\underline{m}\in R} \underline{G}_{12}, \quad (4.19)$$

$$G_{23}^{+} = \max_{m \in R} \overline{G}_{23}, \quad G_{13}^{+} = \max_{m \in R} \overline{G}_{13}, \quad G_{12}^{+} = \max_{m \in R} \overline{G}_{12}. \tag{4.20}$$

imple calculation shows

$$G_{23}^{-} = G_{13}^{-} = G_{12}^{-} = (\theta_1 (2\mu_1)^{-1} + \theta_2 (2\mu_2)^{-1})^{-1} \equiv h, \tag{4.21}$$

$$G_{23}^{+} = G_{13}^{+} = G_{12}^{+} = \theta_1 2\mu_1 + \theta_2 2\mu_2 = u. \tag{4.22}$$

(4.21) and (4.22) are the harmonic mean and arithmetic mean of the shear li respectively.

then have

eorem 4.2. *Given an orthotropic composite made from C_1, C_2 in propor- θ_1, θ_2 then upper and lower bounds on each modulus is given by*

$$E_i^{-} \leqslant E_i^{e} \leqslant E_i^{+}, \text{ for } i = 1, 2, 3, \tag{4.23}$$

$$h \leqslant G_{12}^{e} \leqslant u, \tag{4.24}$$

$$h \leqslant G_{13}^{e} \leqslant u, \tag{4.25}$$

$$h \leqslant G_{23}^{e} \leqslant u. \tag{4.26}$$

›licit algebraic forms for E_i^{-}, and E_i^{+} require tedious computation, however, ‹en component moduli and volume fractions the bounds E_i^{-}, E_i^{+} can be com- numerically. The bounds (4.24)–(4.26) were found origionally in (Lipton & rup 1993). These bounds are saturated by rank 1 orthotropic laminates. In- he upper bound in (4.24) is attained for a rank 1 stiff orthotropic laminate ayer normal parallel to (0, 0, 1). The lower bound in (4.24) is saturated by a l soft laminate with layer normal parallel to (1, 0, 0), or (0, 1, 0). Physically ing, we observe that among all orthotropic composites for a given in-plane the rank 1 stiff laminate with layer normal perpendicular to the shear plane ie strongest resistance to shear. While rank 1 soft laminates with layer nor- parallel to displacement fields associated with in-plane shearing have the t response.

complete the discussion we provide formulas for the remaining off diagonal s of the effective compliance tensors. For stiff laminates these are given by

$$\begin{aligned} -\overline{\nu}_{12}/\overline{E}_2 = d_2 + \theta_1 [c_2^2 \tilde{A}_{12}^2 + d_2^2 (\tilde{A}_{12}^2 + \tilde{A}_{13}^2 + \tilde{A}_{23}^2 + \tilde{A}_{33}^2) \\ + d_2 c_2 (\tilde{A}_{11}^2 + \tilde{A}_{22}^2 + \tilde{A}_{23}^2 + \tilde{A}_{13}^2)], \end{aligned} \tag{4.27}$$

$$\begin{aligned} -\overline{\nu}_{13}/\overline{E}_3 = d_2 + \theta_1 [c_2^2 \tilde{A}_{13}^2 + d_2^2 (\tilde{A}_{12}^2 + \tilde{A}_{13}^2 + \tilde{A}_{22}^2 + \tilde{A}_{23}^2) \\ + d_2 c_2 (\tilde{A}_{11}^2 + \tilde{A}_{12}^2 + \tilde{A}_{23}^2 + \tilde{A}_{33}^2)], \end{aligned} \tag{4.28}$$

$$\begin{aligned} -\overline{\nu}_{23}/\overline{E}_3 = d_2 + \theta_1 [c_2^2 \tilde{A}_{23}^2 + d_2^2 (\tilde{A}_{11}^2 + \tilde{A}_{13}^2 + \tilde{A}_{12}^2 + \tilde{A}_{23}^2) \\ + d_2 c_2 (\tilde{A}_{12}^2 + \tilde{A}_{22}^2 + \tilde{A}_{13}^2 + \tilde{A}_{33}^2)], \end{aligned} \tag{4.29}$$

›ft laminates the entries are:

$$\begin{aligned} -\underline{\nu}_{12}/\underline{E}_2 = d_1 - \theta_2 [c_1^2 \tilde{A}_{12}^1 + d_1^2 (\tilde{A}_{12}^1 + \tilde{A}_{13}^1 + \tilde{A}_{23}^1 + \tilde{A}_{33}^1) \\ + d_1 c_1 (\tilde{A}_{11}^1 + \tilde{A}_{22}^1 + \tilde{A}_{23}^1 + \tilde{A}_{13}^1)], \end{aligned} \tag{4.30}$$

$$\begin{aligned} -\nu_{13}/\underline{E}_3 = d_1 - \theta_2 [c_1^2 \tilde{A}_{13} + d_1^2 (\tilde{A}_{12}^1 + \tilde{A}_{13}^1 + \tilde{A}_{22}^1 + \tilde{A}_{23}^1) \\ + d_1 c_1 (\tilde{A}_{11}^1 + \tilde{A}_{12}^1 + \tilde{A}_{23}^1 + \tilde{A}_{33}^1)], \end{aligned} \tag{4.31}$$

$$-\underline{\nu}_{23}/\underline{E}_3 = d_1 - \theta_2[c_1^2\tilde{A}_{23} + d_1^2(\tilde{A}_{11}^1 + \tilde{A}_{13}^1 + \tilde{A}_{12}^1 + \tilde{A}_{23}^1) + d_1c_1(\tilde{A}_{12}^1 + \tilde{A}_{22}^1 + \tilde{A}_{13}^1 + \tilde{A}_{33}^1)]. \quad (4.32)$$

Here the elements $\tilde{A}_{ij}^s$, $s = 1, 2$ are rational functions of the moments m_i, $i = 1, \ldots, 6$ and are given in the Appendix.

The compliance matrices for orthotropic finite rank laminates are of the form (1.2) and their entries are given through the equations (4.1)–(4.6) and (4.27)–(4.29) for stiff laminates and by (4.7)–(4.12), and (4.30)–(4.32) for soft laminates. We denote the associated matrices by $\overline{C}^{-1}(\underline{m})$ and $\underline{C}^{-1}(\underline{m})$.

For a given ensemble of constant stresses $\sigma^1, \sigma^2, \ldots, \sigma^n$ the bounds on the effective compliance energy $\sum C^{e^{-1}}\sigma^i : \sigma^i$ are given by

$$\min_{\underline{m}\in R} \sum_{i=1}^{n} \underline{C}^{-1}(\underline{m})\sigma^i : \sigma^i \leqslant \sum C^{e^{-1}}\sigma^i : \sigma^i \leqslant \max_{\underline{m}\in R} \sum \overline{C}^{-1}(\underline{m})\sigma^i : \sigma^i. \quad (4.33)$$

For fixed values of component moduli and volume fraction these bounds can be computed numerically.

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Appendix A.

We provide here, the formulas for the functions $\tilde{A}_{ij}^s$ appearing in equations (4.1)–(4.12) and (4.27)–(4.32). Introducing the parameters $\Delta\kappa = \kappa_2 - \kappa_1, \Delta\mu = \mu_2 - \mu_1$ and

$$N^s = \tfrac{1}{9}((\Delta\kappa)^{-1} + 3(\Delta\mu)^{-1}) + \theta_2\tfrac{1}{3}(\tfrac{1}{3}\kappa_s^{-1} + \mu_s^{-1}), \quad s = 1, 2,$$
$$L^s = \tfrac{1}{9}((\Delta\kappa)^{-1} - 3/2(\Delta\mu)^{-1}) + \theta_2\tfrac{1}{3}(\tfrac{1}{3}\kappa_s^{-1} - \tfrac{1}{2}\mu_s^{-1}), \quad s = 1, 2,$$

and the functions t_{11}^s, t_{12}^s, t_{13}^s, t_{22}^s, t_{23}^s, t_{33}^s, $s = 1, 2$, given by (3.3) we define the determinant

$$\begin{aligned}\Delta_1 = {} & (\theta_1 t_{11}^1 + N^1)[(\theta_1 t_{22}^1 + N^1)(\theta_1 t_{33}^1 + N^1) - (\theta_1 t_{23}^1 + L^1)^2] \\ & -(\theta_1 t_{12}^1 + L^1)[(\theta_1 t_{12}^1 + L^1)(\theta_1 t_{33}^1 + N^1) - (\theta_1 t_{23}^1 + L^1)(\theta_1 t_{13}^1 + L^1)] \\ & +(\theta_1 t_{13}^1 + L^1)[(\theta_1 t_{12}^1 + L^1)(\theta_1 t_{23}^1 + L^1) - (\theta_1 t_{13}^1 + L^1)(\theta_1 t_{22}^1 + N^1)],\end{aligned}$$

and the functions $\tilde{A}_{ij}^1$ are defined by

$$\tilde{A}_{ij}^1 = A_{ij}^1/\Delta_1, \quad 1 \leqslant i \leqslant j \leqslant 3$$

where

$$\begin{aligned}A_{11}^1 &= [(\theta_1 t_{22}^1 + N^1)(\theta_1 t_{33}^1 + N^1) - (\theta_1 t_{23}^1 + L^1)^2], \\ A_{12}^1 &= -[(\theta_1 t_{12}^1 + L^1)(\theta_1 t_{33}^1 + N^1) - (\theta_1 t_{23}^1 + L^1)(\theta_1 t_{13}^1 + L^1)], \\ A_{13}^1 &= [(\theta_1 t_{12}^1 + L^1)(\theta_1 t_{23}^1 + L^1) - (\theta_1 t_{22}^1 + N^1)(\theta_1 t_{13}^1 + L^1)], \\ A_{23}^1 &= -[(\theta_1 t_{11}^1 + N^1)(\theta_1 t_{23}^1 + L^1) - (\theta_1 t_{12}^1 + L^1)(\theta_1 t_{13}^1 + L^1)], \\ A_{22}^1 &= [(\theta_1 t_{11}^1 + N^1)(\theta_1 t_{33}^1 + N^1) - (\theta_1 t_{13}^1 + L^1)^2], \\ A_{33}^1 &= [(\theta_1 t_{11}^1 + N^1)(\theta_1 t_{22}^1 + N^1) - (\theta_1 t_{12}^1 + L^1)^2].\end{aligned}$$

oducing the determinant

$$= (N^2 - \theta_2 t_{11}^2)[(N^2 - \theta_2 t_{22}^2)(N^2 - \theta_2 t_{33}^2) - (L^2 - \theta_2 t_{23}^2)^2]$$
$$-(L^2 - \theta_2 t_{12}^2)[(L^2 - \theta_2 t_{12}^2)(N^2 - \theta_2 t_{33}^2) - (L^2 - \theta_2 t_{23}^2)(L^2 - \theta_2 t_{13}^2)]$$
$$+(L^2 - \theta_2 t_{13}^2)[(L^2 - \theta_2 t_{12}^2)(L^2 - \theta_2 t_{23}^2) - (N^2 - \theta_2 t_{22}^2)(L^2 - \theta_2 t_{13}^2)],$$

nctions $\tilde{A}_{ij}^2$ are defined by

$$\tilde{A}_{ij}^2 = A_{ij}^2/\Delta_2, \quad 1 \leqslant i \leqslant j \leqslant 3,$$

$$A_{11}^2 = [(N^2 - \theta_2 t_{22}^2)(N^2 - \theta_2 t_{33}^2) - (L^2 - \theta_2 t_{23}^2)^2],$$
$$A_{12}^2 = -[(L^2 - \theta_2 t_{12}^2)(N^2 - \theta_2 t_{33}^2) - (L^2 - \theta_2 t_{13}^2)(L^2 - \theta_2 t_{23}^2)],$$
$$A_{13}^2 = [(L^2 - \theta_2 t_{12}^2)(L^2 - \theta_2 t_{23}^2) - (L^2 - \theta_2 t_{13}^2)(N^2 - \theta_2 t_{22}^2)],$$
$$A_{23}^2 = -[(N^2 - \theta_2 t_{11}^2)(L^2 - \theta_2 t_{23}^2) - (L^2 - \theta_2 t_{13}^2)(L^2 - \theta_2 t_{12}^2)],$$
$$A_{22}^2 = [(N^2 - \theta_2 t_{11}^2)(N^2 - \theta_2 t_{33}^2) - (L^2 - \theta_2 t_{13}^2)^2],$$
$$A_{33}^2 = [(N^2 - \theta_2 t_{11}^2)(N^2 - \theta_2 t_{22}^2) - (L^2 - \theta_2 t_{12}^2)^2].$$

References

eda, M. 1987 Optimal bounds and microgeometries for elastic composites. *SIAM J. appl.* :. **47**, 1216–1228.

rt, G. & Murat, F. 1986 Homogenization and optimal bounds in linear elasticity. *Arch. n. Mech. Analysis* **94**, 307–334.

kii, L. & Cherkaev, A. V. 1984 Design of composite plates of extremal rigidity. Ioffe icotechnical Institute. Preprint (in Russian).

Z. & Shtrikman, S. 1962 A variational approach to the theory of the elastic behaviour ıltiphase materials. *J. Mech. Phys. Solids* **11**, 127–140.

Z. & Rosen, B. 1964 The elastic modulus of fiber-reinforced materials. *ASME J. appl.* .. **31**, 223.

Z. 1965 On elastic behaviour of fibre reinforced materials of arbitrary transverse phase etry. *J. Mech. Phys. Solids* **13**, 119.

D. 1888 Darstell, definirter formen als summe von formenquadraten. *Mathematische :len* **32**, 342–350.

1964 Theory of mechanical properties of fibre-strengthened materials. 1. Elastic beur. *J. Mech. Phys. Solids* **12**, 199.

R., Lipton, R. & Lutoborski, A. 1990 Laminar elastic composites with crystallographic ıetry *SIAM J. appl. Math.* **50**, 683–702.

:. V. & Lipton, R. 1988 Optimal bounds for the effective energy of a mixture of isotropic, ıpressible elastic materials. *Arch. ration. Mech. Analysis* **102**, 331–350.

A. G. & Nudel'man, A. A. 1977 *The Markov moment problem and extremal problems.* :lations of Mathematical Monographs, vol. 50. AMS.

R. 1986 An optimal lower bound on the energy dissipation rate for homogenized Stokes Ph.D. thesis, New York University.

R. 1992 Bounds and perturbation series for incompressible elastic composites with verse isotropic symmetry. *J. Elasticity* **27**, 193–225

R. 1991 On the behaviour of elastic composites with transverse isotropic symmetry. *J. . Phys. Solids* **39**, 663–681.

Lipton, R. & Northrup, J. 1993 Bounds on shear moduli for orthotropic composites. In *Proc. 1993 North American Conference on Smart Structures and Materials, Mathematics in Smart Structures* (ed. H. T. Banks).

Lurie, K. A. & Cherkaev, A. V. 1984 Exact estimates of conductivity of composites formed of two isotropically conducting media taken in prescribed proportion. *Proc. R. Soc. Edinb.* A**99**, 71.

Milton, G. W. & Kohn, R. V. 1988 Variational bounds on the effective elastic moduli of anisotropic composites. *J. Mech. Phys. Solids* **36**, 597–629.

Milton, G. W. 1985 *Communs math. Phys.* **99**, 465.

Norris, A. 1985 A differential scheme for the effective moduli of composites. *Mech. Mater.*4, 1.

Tartar, L. 1985 Estimations fines de coefficients homogeneises. In *Ennio De Georgi Colloquium* (ed. P. Kree), pp. 168–187. Research Notes in Mathematics 125. Boston: Pitman.

Thurston, R. 1984 Waves in solids. In *Mechanics of solids* (ed. C. Trusdell), vol. 4. Berlin and New York: Springer-Verlag.

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